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Integrating VMI into joint replenishment planning for optimized manufacturing supply chains

Bassem Roushdy[a*]

[a]*Basic and Applied Science Department, College of Engineering and Technology, Arab Academy for Science, Technology, and Maritime Transport, Cairo, Egypt*

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This paper presents a new integrated framework combining the Joint Replenishment Problem (JRP) with a generalized Vendor Managed Inventory (VMI) system. The model under consideration represents a three-level supply chain consisting of a supplier, manufacturer, and retailer. The model incorporates multiple product types, each produced on a dedicated machine at the manufacturer, subject to setup costs, and major and minor ordering costs. The primary objective of this research is to optimize a set of critical decision variables, including the common order interval, order frequencies for each item, backorder levels at the retailer, and production initiation times at the manufacturer for each product type, under both deterministic and stochastic demand scenarios. This analysis will provide valuable insights for improving joint replenishment operations in manufacturing. The research begins with a deterministic model fit for the particular issue area derived from the canonical JRP. Within a VMI context, the manufacturer, acting as the supply chain leader, utilizes shared information to derive initial feasible solutions. Subsequently, an optimization technique is employed, combining marginal cost-based and cumulative cost-based algorithms, while leveraging embedded discrete Markov chain decomposition method adapting Jacobi stepping method to determine steady-state probabilities. A cost function is then defined for each action state within this framework. The integration of the VMI policy into the JRP model can significantly reduce the whole cost of the supply chain, through balancing between production initiation and backorders under both the deterministic and stochastic models.

1. Introduction

Effective inventory management is essential to supply chain operations in today's corporate environment. The VMI paradigm optimizes inventory levels and replenishment cycles throughout the whole supply chain, from raw materials to completed goods, providing a strategic solution to the complex problem of joint replenishment. The VMI approach is a game-changer, simplifying supply chains and tackling complicated replenishment difficulties from start to finish. By giving the manufacturer control over inventory management, VMI increases productivity (Olhager & Wästlund, 2018). Many studies underline how important supply chain agility is for flexibility and cost control (Chopra & Meindl, 2024; Christopher, 2016). A strong alliance, a whole awareness of the supply chain, and efficient inventory contro are crucial with careful regard to order quantities, joint replenishment can enhance inventory performance even with transportation charges (Musalem & Dekker, 2005). Good inventory control requires thinking through production limits to create strong replenishment plans that lower costs and raise service standards (Schouten et al., 2021; Choi & Lambert, 2016).

Using information sharing,

By the exchange of current sales and inventory information, suppliers develop a better insight into the actual patterns of demand, allowing them to strategize their supply chain activities accordingly. This allows for greater supply chain flexibility

* Corresponding author
E-mail address bassem.roushdy@aast.edu (B. Roushdy)

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through the empowerment of decision-making through data on the allocation of resources, reconfiguring manufacturing plants, and quick reaction to market changes (Koot et al., 2021).

For purposes of injecting realism and pragmatism to supply chain models, it is necessary to incorporate a series of resource limits. Such resource limits include minimum order quantities, transportation capacity constraints, budget limitations, as well as production capacity, demand uncertainty, and production capacity (Liu, 2023). One of the operational problems that encompass the cost savings in inventory is the coordination of items orders to have cost savings resulting from consolidated orders. In the classical Joint Replenishment Problem (JRP), the order interval for each item should be an integer multiple of a base period, hence making the simultaneous ordering of more than one item feasible. The quantity ordered by each item for a period of time is represented by a coefficient unique to the item's ordering cycle in the base period, (Silver 1979). Researchers have developed various mathematical models to capture the intricacies of multi-item inventory systems, aiming to minimize total supply chain costs while considering diverse constraints. To address the computational complexity of the JRP, numerous studies have employed heuristic approaches, such as genetic algorithms (Moon & Cha, 2006; Yoo & Gen, 2007; Hong & Kim, 2009; Taleizadeh et al., 2013) and integer programming (Levi et al., 2006; Kang et al., 2013). One significant advancement in JRP modeling involves incorporating stochastic demand, as evidenced in the work of Minner & Silver (2007), Feng et al. (2015), and Kayiş et al. (2008). By acknowledging the inherent variability of real-world demand, these models consider key decision variables such as order quantities, reorder points, and replenishment intervals for each item.

Previous studies on three-echelon supply chains frequently neglect the complexities of manufacturing operations within the JRP framework. Research conducted by Ben-Daya et al. (2013), Sana et al. (2014), and Büyükkaramikli et al. (2013) addresses three-echelon systems but tends to simplify the manufacturing processes at the manufacturer level. Additionally, while numerous researchers have explored Vendor Managed Inventory (VMI) models for two-layer supply chains involving multiple products (such as Vlist et al., 2007; Darwish & Odah, 2010; Yu et al., 2012; Mateen & Chatterjee, 2015; Mateen et al., 2015; Taleizadeh et al., 2015a; Khan et al., 2016), they have not examined the consolidation of items to create a common timeframe.

Building upon Banerjee's (1986) work, which emphasized the significance of limited production capacity in vendor-buyer interactions, Nobles (2022) introduced a more sophisticated model. This research analyzes a finite-capacity, two-product system with stochastic order quantities and inter-order times, reflecting real-world complexities. The study investigates (s, S) and (s, c, S) replenishment policies within a framework that models the production facility as a multi-type MMAP[K]/PH[K]/1 queue. Matrix analytic methods are employed for analysis, with the model explicitly incorporating stochastic nature. Previous research has indicated that can-order policies may exhibit inferior performance compared to periodic joint replenishment policies in scenarios where major setup costs are significant (Atkins & Iyogun, 1988; Pantumsinchai, 1992; Viswanathan, 1997, 2007).

Detailed Contribution:

This study departs from Classical JRP models, which often concentrates on multi-supplier scenarios, by analyzing a production-oriented JRP with a single source (the manufacturer). Key decision variables include optimal start time for production, backorder levels, common cycle length, and order frequencies between several items, all in an environment where demand is uncertain and the production capacity is finite.

VMI implemented in the supply chain strategy in order to fosters collaboration among the suppliers and the retailers with improved inventory management and cost reduction. The manufacturer assumes the responsibility of managing the mutual replenishment of raw materials from the supplier to meet the production needs. In the process, he also manages the level of inventory at the retailer by setting a single replenishment cycle and making decisions on what items will be included in the shipment. He also determines the best up to level inventory position for every item, and setting proper backorder with the aim of reducing total system costs.

The deterministic model is initially solved, followed by an extension to incorporate the influence of demand uncertainty. To account for this stochasticity, steady-state inventory levels at retailer are computed using embedded discrete-time Markov chains which monitor the inventory level at definite times. This approach focuses on specific time points: before the initiation of production, at the shipping time, and following the shipment of products. Subsequently, a Markov decision process is employed to identify the optimal state-action pairs and minimize the associated costs.

The system dynamics are described in the subsequent section. This is followed by the introduction of two policies under deterministic conditions, which are then extended to accommodate stochastic scenarios. Finally, the numerical results and conclusions are presented.

System Dynamics:

Extending the foundational work of Taleizadeh (2020), the model investigates a three-echelon supply chain comprising a raw material supplier, a manufacturer, and a retailer. The manufacturer produces multiple product types $i, i \in [1, n]$, each on a distinct production line characterized by finite production capacity with production rate p_i units per unit time and associated setup cost b_i and a raw material order cost c_i for inventory replenishment. Inventory holding costs h_{ji}, expressed per unit time, are incurred for each item i at each location j, ($j \in [1 ,2\ 3]$; 1: retailer, 2: finished products, 3: raw materials). Customer demand per unit time d_i occurs exclusively at the retailer level, with backorders permitted at a cost of Sh_i per unit per unit time. Machine utilization factor ρ_i equals to d_i / p_i and should be less than one for system stability.

Within this framework, there are major order costs (A & B) incurred by both the retailer and the manufacturer for each order placed from each location respectively. This cost encompasses the transportation and dispatching expenses associated with the entire order. Moreover, each individual item within the order for the retailer is subject to a minor order cost of a_i. A key assumption is that the replenishment cycles for all three echelons are integer multiples of a common time interval (T) and the manufacturer orders raw materials from a supplier with an ample supply.

Under VMI, the supplier assumes responsibility for replenishment decisions, optimizing inventory levels at the retailer's location. A considerable body of research has explored VMI models within two-layer supply chains (e.g., Von Stackelberg, 2011 Vlist et al., 2007; Darwish & Odah, 2010; Yu et al., 2012; Mateen & Chatterjee, 2015; Taleizadeh et al., 2015a; Khan et al., 2016)

In our case, the manufacturer assumes the role of supply chain leader, actively monitoring inventory levels for items at the retailer. To maximize efficiency, the manufacturer strategically schedules production to replenish multiple items in a single shipment. This coordinated approach allows the retailer to capitalize on economies of scale by consolidating various items into a single major order.

This research focuses on minimizing total supply chain costs by coordinating raw material replenishment, final product delivery, and production activities. In the deterministic model, key decision variables include:

- The common cycle time (T)
- The order frequency (k_i) for each item i at the retailer level, representing the number of replenishments per cycle.
- The number of retailer replenishment cycles (n_i) covered by a single production run.
- The order frequency (u_i) for raw material replenishment for each item i at the manufacturer level.
- The permissible production delays(y_i) for each item at the manufacturer .These variables are crucial for optimizing the coordination of activities across the supply chain.
- In the stochastic case, the order up to level S_i for the retailer and safety stock (SS_i) is introduced as additional decision variables with the above mentioned.

In JRP stochastic environments, the simultaneous optimization of production schedules and inventory levels for many items, while minimizing cost and production capacity constraints, is a formidable computational task. The complexity is due to the existence of a large number of possible steady states, making the problem computationally infeasible. A common approach is to initially adopt a deterministic model, assuming fixed demand patterns. This simplified model provides a starting point for the optimization process. Then, the model is extended to incorporate stochastic demand, thus inventory position and safety stock levels may be determined for each autonomous item in order to deal with uncertainty at minimum cost.

Policy One (Deterministic Case): JPP with VMI without Production Delay and No Shortage

In this model we assume that the manufacturer which paly the primary role in the (VMI-JRP) model, monitors the inventory level at the retailer, server starts production to raise the retailers inventory level to pre specified level S_i ($S_i = k_i\, d_i\, T$) then continues production to cover demand for n_i retailer's cycles. Hence manufacturer's total production quantity now is ($n_i\, k_i\, d_i\, T$) and produced in time($n_i\, k_i\, \rho_i\, T$), as shown in Fig. 1.

The manufacturer places orders with the supplier to replenish its raw material inventory. Each order placed by the manufacturer is designed to cover subsequent manufacturer order cycles. The total quantity of raw materials ordered is ($u_i\, n_i\, k_i\, d_i\, T$), assuming a unit-to-unit correspondence between raw material consumption and finished product production.

The total cost of the three echelon system is derived, as shown in Fig. 1, by first assessing the area representing the total production volume of the manufacturer, without considering any downward shipments. Then the quantity shipped to the retailer at each cycle is subtracted from this total area to arrive at the final cost. Since demand is deterministically known, there is neither back order at retailer nor leftover quantities at manufacturer.

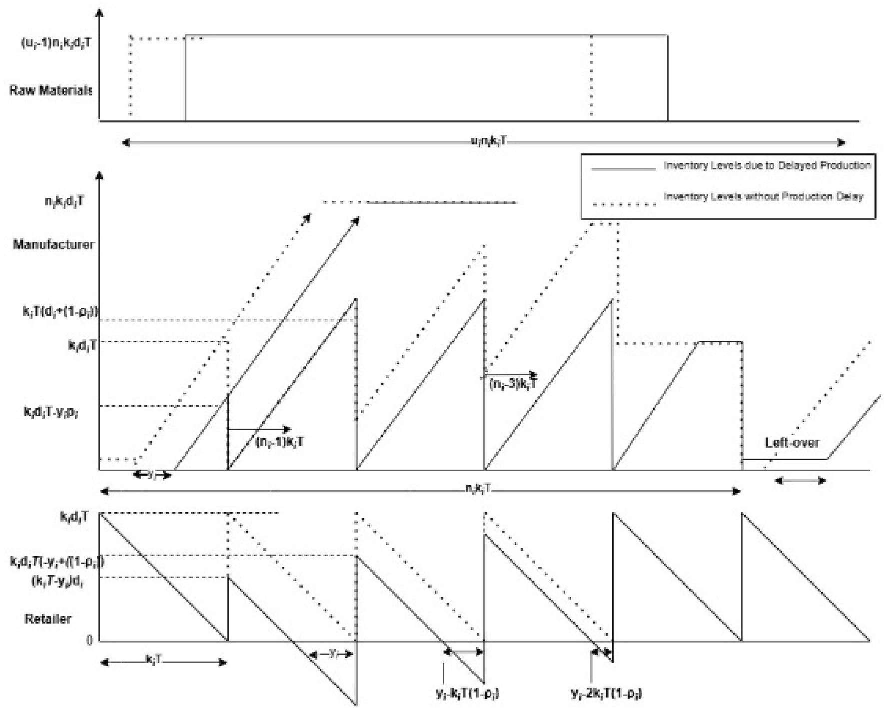

Fig. 1. Presents Inventory Levels for Normal Start time and Delayed Start Time

The determination of the total cost of the three-echelon system commences with the assessment of the total production volume of the manufacturer, as illustrated in Figure 1. Subsequently, the quantity of product shipped to the retailer at each cycle is subtracted from this total volume to arrive at the final cost. Given the assumption of deterministic demand, the model does not incorporate the possibility of backorders at the retailer or the presence of leftover quantities at the manufacturer.

$$TC = \frac{A+B}{T} + TC_r\, \{\textstyle\sum_{i=1}^{n} \emptyset\, (k_i\, T)\} + TC_w\{\textstyle\sum_{i=1}^{n} \emptyset\, (k_i\, , n_i T)\} + TC_s\{\textstyle\sum_{i=1}^{n} \emptyset\, (\, n_i k_i, u_i T)\}$$

$$TC = \frac{A+B}{T} + \sum_{i=1}^{n}\left(\frac{\frac{a_i}{k_i n_i}+\frac{b_i}{k_i n_i}+\frac{c_i}{k_i n_i u_i}}{T} + \frac{k_i d_i T}{2}\big(h_{1i} + h_{2i}\,(n_i - 1 + (2-n_i)\rho_i) + h_{3i}(u_i + \rho_i - 1)\big)\right) \tag{1}$$

To determine the optimal time between order intervals, the multiplier constants are initially set to one. Equation (1) is then differentiated with respect to T and set to zero, resulting in the common time between intervals (T) as shown in Eq. (2).

$$T = \sqrt{\frac{2\left(A+B+\sum_{i=1}^{n}[a_i/k_i + b_i/n_i k_i + c_i/u_i n_i k_i]\right)}{\sum_{i=1}^{n} k_i d_i \partial_i}},\ \partial_i = (h_{1i} + h_{2i}(n_i - 1 + (2-n_i)\rho_i + h_{3i}(u_i - \rho_i - 1) \tag{2}$$

Subsequently, after determining the optimal value of T, the junction point method, proposed by Wildeman et al. (1997) to solve the JRP model, is utilized to ascertain the value of $[k_i, n_i]$ for specific value of T,

$$T\,(k) \le T(k-1),\ T\,(k+1) \le T(k), \tag{3}$$

where

$$-\frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{8(n_i u_i a_i + u_i b_i + c_i)}{u_i n_i T^2 d_i(\partial_i)}} \ \le\ k_i \ <\ \frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{8(n_i a_i + b_i)}{u_i n_i T^2 d_i(\partial_i)}}.$$

$$T\,(n,k) \le T(n-1,k), \qquad T\,(n+1,k) \le T(n,k), \tag{4}$$

where

$$-\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8(u_i b_i+c_i)}{u_i n_i T^2 d_i(\partial_i)}} \leq n_i < \frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{8(n_i a_i+b_i)}{k_i T^2 d_i(\partial_i)}}.$$

The multiple integer value for the supplier is computed by employing the next equation, which incorporates the specified multiple integer values for the retailer and manufacturer

$$u_i = \frac{\sqrt{2c_i/h_{3i}d_i}}{k_i n_i T}. \tag{5}$$

The values of the multipliers $[k_i, n_i \& u_i]$ are approximated to the nearest integer using the nearest integer method. These integer values are subsequently substituted back into Eq. (2). This iterative process continues until convergence is achieved, with convergence typically observed within three to four iterations.

Policy Two: JPP with VMI with Production Delay and Backorders

In Policy 1, production for each item commences at time prior to shipment by $(k_i T - k_i \rho_i T)$, enabling the manufacturer to deliver the required quantity to replenish the retailer's inventory to the desired level. Policy 2 introduces a deviation from this approach by delaying the initiation of production at the manufacturer. While this delay results in a reduction in the manufacturer's holding costs, it may also lead to an increase in shortage costs at the retailer due to potential backorders. These backorders are subsequently fulfilled in subsequent production cycles. The raw material order time is correspondingly shifted by the same duration.

Each retailer cycle requires $(k_i \rho_i T)$ units time for production, so accordingly the maximum permissible delay time for the manufacturer to start production to ensure no lost demand is $(n_i k_i (1-\rho_i)T)$. The quantity of backorders at the retailer may be fulfilled within one or more manufacturer cycle, contingent upon the quantity backordered. If item (i) experiences a delay of (y_i) during the manufacturer's initial production cycle, the number of retailer cycles, $\check{c}_i$ affected by this delay is calculated using the following formula: $[\check{c}_i = y_i/(k_i T(1-\rho_i))]$.

Each retailer cycle necessitates $(k_i \rho_i T)$ units of time for production at the manufacturer. Consequently, the maximum permissible delay in the initiation of production to ensure the avoidance of lost demand is $(n_i k_i (1-\rho_i)T)$. If item (i) experiences a production delay of (y_i) , then accordingly number of retailer cycles within single manufacturer cycle, $\check{c}_i$, with backorders affected by this delay is $\check{c}_i = y_i/(k_i T(1-\rho_i))$

Following an approach similar to Pan and Yang (2002), the total cost (TC_i) incurred for each item (i) within this system is calculated as follows:

$$TC_i = \frac{\frac{a_i}{k_i}+\frac{b_i}{k_i n_i}+\frac{c_i}{k_i n_i u_i}}{T} + \left(\frac{n_i-\check{c}_i}{n_i}\right)\frac{h_{1i}k_i d_i T}{2} + \frac{\check{c}}{n_i}\left(\frac{h_{1i}\sum_{i=1}^{\check{c}}(k_i T+(i-1)k_i T(1-\rho_i)-y)^2 d_i}{2k_i T} + \pi_i \sum_{i=1}^{\check{c}_i}\big(y_i-(i-1)k_i T(1-\rho_i)\big)^2 \frac{d_i}{2}\right) + h_{2i}\left[\frac{k_i d_i T}{2}\,(n_i-1+(2-n_i)\rho_i) + y_i\left(\frac{p_i(n_i-1)}{n_i}-d_i\right) - \left(\sum_{i=1}^{\check{c}_i}(k_i T(1-\rho_i)p_i\left(\frac{n_i-i-1}{n_i}\right)\right) + \left((\check{c}_i\, k_i T(1-\rho_i)-y_i)p_i\left(\frac{n_i-\check{c}_i-1}{n_i}\right)\right)\right] + h_{3i}\left[\frac{k_i d_i T}{2}(u_i+\rho_i-1)\right] \tag{6}$$

The first term represents the aggregate minor ordering costs and setup costs incurred within the system. The second term accounts for the holding costs associated with retailer cycles that remain unaffected by the production delay. The third term encompasses the summation of holding costs and shortage costs per unit time for those retailer cycles that are affected by the production delay (y_i), where the delay time is gradually reduced by $k_i T(1-\rho_i)$ units per cycle, and the backorders are reduced by $k_i T(1-\rho_i)p_i$ units each cycle by The fourth term represents the holding costs incurred by the manufacturer due to the production delay. Finally, the fifth term accounts for the holding costs associated with the raw materials inventory.

The optimal value of (y_i) is constrained within a range bounded by a minimum of zero and a maximum value of $n_i k_i T(1-\rho_i)$, Employing a dichotomous search method, the delay time for each machine at the manufacturer is determined. Subsequently, the optimal value of (T) is obtained through differentiation of the aforementioned cost function, resulting in the following equation:

$$\sqrt{\frac{2*\left(A+B+\sum_{i=1}^{n}\left[\frac{a_i}{k_i}+\frac{b_i}{k_i n_i}+\frac{c_i}{k_i n_i u_i}+\frac{h_{1i}\check{c}_i d_i y_i^2}{2k_i n_i}+\frac{s\check{c}_i d_i y_i^2}{2k_i n_i}\right]\right)}{\sum_{i=1}^{n}\left(k_i d_i\left(\partial_i+\frac{\pi_i\check{c}_i\left(1-3\check{c}_i+2\check{c}_i^2\right)(-1+\rho_i)^2}{6n_i}+\frac{h_{1i}\left(6n_i+2\check{c}_i^3(-1+\rho_i)^2-3\check{c}_i^2(-1+\rho_i{}^2)+\check{c}_i(-5+4\rho_i+\rho_i{}^2)\right)}{6n_i}\right)+\frac{h_{2i}(-1+\check{c}_i)\check{c}_i k_i p_i(-1+\rho_i)}{n_i}\right)}} \tag{7}$$

The optimization process proceeds as follows:

1) The common cycle time is initially computed using equation (2).
2) The multiplier constants are then calculated using equations (3), (4), and (5), respectively.
3) These newly calculated multiplier constants are subsequently substituted back into equation (2) to obtain an updated cycle time.
4) This iterative process continues until the difference between successive cycle time calculations falls below a specified tolerance level, indicating convergence
5) The upper limit of the delay time is determined, and a search procedure is employed to identify the optimal value of y that minimizes equation (6).
6) The optimal value of (y_i) is then substituted back into equation (7) to obtain a new common cycle time.
7) The multiplier values are initialized to one, and their values are incrementally increased until the cost function increases.
8) The common cycle time is recalculated using equation (7), and the value of (y_i) is re-determined. This iterative process continues until convergence of T and the total cost.

Stochastic Case: Policy One (SP1): Uncertain Demand, No Production Delay

To adapt the deterministic model to stochastic factors and to determine its steady-state system performance, each item is modeled as a discrete-time embedded Markov chain. The state transitions of this Markov chain occur at the time of production initiation at the manufacturer and the time at which the ordered quantity is shipped to the retailer. The arrival process of customer demands is assumed to follow a Poisson distribution with intensity $\lambda_i\ i \in N; N = \{1, 2, .. i .. , N\}$. Within the VMI-JRP system, a $(k_i T, S_i)$ inventory policy is implemented at retailer for each item. This policy dictates that at fixed review intervals, inventory levels are checked, and an order is placed to bring the inventory position (on-hand inventory + outstanding orders - backorders) back to a predetermined target level. This policy is extensively discussed in Axsäter (2006) and Zipkin (2000).

The manufacturer initiates production when its inventory level becomes insufficient to fulfill the upcoming order requirements from the retailer. In accordance with the JRP model, items are consolidated for shipment to the retailer to facilitate the equitable distribution of major order costs. At the designated time interval (T) with zero lead time, all items included within the shipment are delivered to the retailer, with the objective of raising the retailer's inventory level to S_i. However, in the initial production cycle, the quantity of items produced may not be sufficient to achieve the pre-specified inventory level, potentially resulting in backorders. In subsequent production cycles, assuming no production delays, the manufacturer is expected to attain the desired inventory level at the retailer. Conversely, the production quantity at the end of the production period may exceed the retailer's demand, resulting in excess inventory (o_i) at the manufacturer.

Since the individual retailer cycles within a given manufacturer cycle are not identical. Each distinct retailer cycle within a manufacturer cycle is uniquely identified by the index $(z_i, z_i = 1,2..n_i)$ where $z_i = 1$ denotes the first retailer cycle and denotes & n_i the last retailer cycle.

The analysis employs an embedded discrete-time Markov chain approach to capture steady-state probabilities at two critical time points:

- **Production Initiation:** Inventory levels $IL(z)_i^p$ are evaluated prior to the commencement of production for each item, considering a preparation time of units for each item.
- **Shipment to Retailer:** Inventory levels at the retailer are evaluated both before and after the shipment of goods $IL(z)_i^e, IL(z)_i^s$ respectively. This analysis enables the determination of key performance indicators such as the amount of inventory shipped to the retailer, the amount of leftover inventory at the manufacturer at the end of its cycle $\hat{a}(z)_i$, the optimal safety stock level $SS(z)_i$, the number of backorders $B(z)_i$, and the cycle time (T)

Transition and Steady State Probabilities

Our methodology deviates from conventional approaches by exclusively considering certain states. These states represent the system state immediately subsequent to the placement of an order and the production initiation. The Retailer's inventory levels fluctuate based on the decisions made regarding the timing of manufacturer production initiation, demand uncertainty and the common cycle time (T). The analysis is restricted to transitions between these states, for which a Discrete-Time Markov Chain (DTMC) is established. Let $IL(z)_i^p = \{ (IL(z)_i^p = k) \, \forall \{k = 0 \; to \; S_i\}$ denote the steady-state distribution of retailer's inventory at time of production, $IL(z)_i^e = \{ (IL(z)_i^e = k) \, \forall \{k = 1 \; to \; S_i\}$ denotes the steady state retailer's inventory immediately before shipment and $IL(z)_i^e = \{ (IL(z)_i^s = k) \, \forall \{k = 1 \; to \; S_i\}$ denotes the steady state retailer's inventory immediately after shipment. The determination of these probabilities is achieved through an iterative process. This iterative process commences with an initial condition where the retailer inventory level $IL(1)_i^p$ at the time of production is assumed to be $k_i(1-\rho_i)d_iT$ with probability one. Then the process continues as follows:

$$P(IL(1)_i^e = k) = \sum_{j=0}^{S}(P(IL(1)_i^p = j) * P(D(t) = j - k) = \sum_{j=0}^{S}(P(IL(1)_i^p = j) * \frac{\lambda t^{IL(1)_i^p - k}}{(IL(1)_i^p - k)!} e^{-\lambda_i t'} \tag{8}$$

where $t' = k_i\rho_iT + E(W_{1i})$ and $E(W_{1i}) = E(IL(1)_i^{p^-})/\lambda_i$. Since the production rate is constant, the retailer's inventory level at the start of its cycle is the same as the inventory level at the end of its previous cycle, adjusted by the amount of inventory shipped from the manufacturer.

$$(P(IL(z)_i^s) = k + k_i\rho_id_iT + B(z)_i - o_i) = (P(IL(z)_i^e\,) = k) \qquad \{ z_i = 1,2..n_i\} \tag{9}$$

$$P\big(IL(1)_i^p = k\big) = \sum_{j=0}^{S}(P(IL(1)_i^s = j) * P(D(t) = j - k) = \sum_{j=0}^{S}(P(IL(1)_i^s = j) * \frac{\lambda t^{IL(1)_i^s - k}}{(IL(1)_i^s - k)!} e^{-\lambda_i t''} \tag{10}$$

where $t'' = k_i\rho_iT + E(W_{oi})$, $E(W_{oi}) = E(IL(1)_i^{s^-})/\lambda_i$ and $[\,E(W_{oi})\,\&\,E(W_{1i})\,]$ represent the expected waiting time resulting from negative inventory levels $(E(IL(1)_i^{s^-})\,\&\,E(\,IL(1)_i^{p^-}))$ respectively. Although theses probabilities are very low, it is crucial to incorporate them into the system to ensure a stable system that accommodates backorders. To evaluate the waiting time, we exploit the fact *that* $E(IL^-) = E(IL) - E(IL^+)$

Model Execution Steps:

1. **Initialization:**

   The initial condition for the DTMC is the time when the retailer's inventory level reaches the level m_i, which is equal to $k_i(1-\rho_i)d_iT$ units.

2. **Steady-State Probability Calculation (First Iteration):**
   - In the first iteration, steady-state probabilities $(IL(z)_i^e)$ are evaluated using $P(IL_i^e(1) = k) = P(D(t) = m_i - k) = \frac{\lambda t^{m_i - k}}{(m_i - k)!} e^{-\lambda_i t}$
   - Since there are no initial leftover inventories or backorders, the amount shipped equals the production quantity.
   - $IL(1)_i^e$ is equivalent to $P(IL(z)_i^s = k + k_i\rho_id_iT) = P(IL(1)_i^e = k)$, as adjusted by the amount produced $k_i\rho_id_iT$.
3. **Handling Negative Inventory:**
   - To ensure system stability and a finite space states within the model, the infinite negative inventory levels $P(IL(z)_i^s \leq 0)$ & $P\big(IL(1)_i^p \leq 0\big)$ are combined into the expected mean demand as waiting time.
   - The effect of negative inventory is accounted in the expected mean demand as shown in Eq. (8) & (12)
   - In the first iteration, the expected negative inventory $E(IL(z)_i^{s^-})$ equals to $\sum_{i=1}^{S} i * \big(P(IL(z)_i^s = i)\big) - (m(i) + k_id_iT(1-\rho_i)$, $z = \{1 \; or \; 2\}$ and in subsequent iterations it equals to $\sum_{i=1}^{S} i * \big(P(IL(z)_i^s = i)\big) - (\sum_{i=1}^{S} i * \Big(P\big(IL(z)_i^p = i\big)\Big) + k_id_iT(1-\rho_i)$, while $E\big(IL(1)_i^{p^-}\big)$ equals to $\sum_{j=0}^{S} j * (P(IL(1)_i^p = j)) - (\sum_{j=0}^{S} j * (P(IL(1)_i^s = j)) - k_i(1-\rho_i)T - E(IL(1)_i^{s^-}))$.
4. **Backorder Calculation:**
   - Since The retailer's inventory position is (on-hand inventory + outstanding orders - backorders). Accordingly at steady state, the amount shipped equals production plus backorders minus leftover inventory is then evaluated numerically and the left over amount $(\,o_i)$ at steady state is equal to $k_i\rho_id_iT - \sum_{j=m(i)}^{S} j * (P(IL(1)_i^p = j)$

- The backorder amounts per unit time for each cycle $B(z)_i$ are determined as follows: $E(IL(z)_i^{e-}(t)) = \sum_{j=0}^{S} j * (P(IL(z)_i^s = j) - (\lambda_i k_i \rho_i T) - \sum_{i=1}^{S} i * (P(IL(1)_i^e = i))$

5. **Iteration and Convergence:**
   - The outputs from steps 3 and 4 are used as inputs for the next iteration.
   - This iterative process continues until convergence is achieved.
6. **Negative Inventory and Safety Stock:**
   - Negative inventory levels $IL_i^{p-}(z), IL_i^{s-}(z), IL_i^{e-}(z)$ are evaluated using equations (9), (10), and (12) after convergence to obtain accurate estimates for the safety stock.
7. **Production Capacity and Inventory Levels:**
   - With no production delays and excess production capacity, we assume a negligible probability of insufficient production to elevate the subsequent retailer's inventory level ($IL(z)_i^s$) $[z_i = 2 \ to \ n_i]$ to pre specified position S_i .
   - Therefore, $IL(z)_i^e$ and $B(z)_i$ $[z_i = 2 \ to \ n_i]$ are directly calculated using the provided equations, $P(IL(z)_i^e = k) = P(D(t) = j - k) = \frac{\lambda t^{S-k}}{(S-k)!} e^{-\lambda t}$ *and* $B(z)_i = \sum_{j=-\infty}^{0} -j * (P(IL_i^e = j)$

Cost Optimization Procedure

The optimization process is conducted in two phases: First, the safety stocks $SS(z)_i$ for each item are evaluated. Then a search procedure is conducted on the common cycle time (T) to identify the minimum system cost. The multiplier frequencies used in this cost calculation are those determined from the deterministic model.

The determination of optimal safety stock levels involves a careful consideration of the trade-off between the costs associated with holding inventory and the costs incurred due to stock outs. An increase in the inventory level from S_i to $S_i + x$ induces a corresponding shift in the expected inventory levels probabilities in the positive part $(E(IL(z))_i^+ | \, SS_i = x) = \sum_{k=-SS_i}^{S_i+SS_i} (k + x) * p(IL_i = k | \, SS_i = 0)$ and the negative part $(E(IL(z))_i^- | \, x) = \sum_{k=-\infty}^{-SS_i-1} (k + x) * p(IL_i = k | \, SS_i = 0)$.

The expected amount of leftover inventory due to the addition of safety stock amount is calculated in accordance with the following equation: $E(o(i)) = \sum_{k=k_i(1-\rho_i)d_iT+SS_i}^{S_i+SS_i} (SS_i + S_i - k) * p(IL_i^p = k | \, SS_i = 0)$

The optimal level of safety stock for item (i) at the retailer for $z = 1$, denoted by $SS(1)_i$, is determined as $\arg \min_{SS(1)_i} f := \arg \min_{x \in SS(1)_i} f \left(h_{1i} \left(\frac{(E(IL(z))_i^{e+} | \, x) + (E(IL(z))_i^{s+} | x}{2} + \frac{1}{n_i} o_i \right) + \pi_i \left(\frac{(E(IL(z))_i^{e-} | \, x) + (E(IL(z))_i^{s-} | \, x)}{2} \right) \right) \{x \in SS(1)_i : f(SS(1)_i) > f(x) > \forall \, x \in SS(1)_i \}$ and for $SS(z > 1)_i$, by : $\arg \min_{SS(z)_i} f := \arg \min_{x \in SS(z)_i} f \left(h_{1i} \left(\frac{(E(IL(z))_i^{e+} | \, x) + (E(IL(z))_i^{s+} | x)}{2} \right) + \pi_i \left(\frac{(E(IL(z))_i^{e-} | \, x) + (E(IL(z))_i^{s-} | \, x)}{2} \right) \right) \{x \in SS(z)_i : f(SS(z)_i) > f(x) > \forall \, x \in SS(z)_i \}$

The total cost per unit time for the system after determining the optimal $SS(z)_i$ is calculated using the following equation:

$$TC_{sp1} = \frac{A+B}{T} + \sum_{i=1}^{n} \left(\frac{\frac{a_i}{k_i} + \frac{b_i}{k_i n_i} + \frac{c_i}{k_i n_i u_i}}{T} + \left(\frac{1}{n_i}\right) \left[h_{1i} \left(\frac{E(IL(1))_i^{e+} + E(IL(1))_i^{s+}}{2} \right) + \pi_i \left(\frac{E(IL(1))_i^{e-} + E(IL(1))_i^{s-}}{2} \right) + h_{2i} . E(o(i)) \right] + h_{2i} \left[\frac{k_i d_i T}{2} (n_i - 1 + (2 - n_i)\rho_i) \right] + h_{3i} \left[\frac{k_i d_i T}{2} (u_i + \rho_i - 1) \right] + \sum_{z=2}^{n_i} \left[h_{1i} \left(\frac{E(IL(z))_i^{e+} + E(IL(z))_i^{s+}}{2} \right) + \pi_i \left(\frac{E(IL(z))_i^{e+} + E(IL(z))_i^{s+}}{2} \right) \right] \right) \quad (11)$$

In a deterministic Joint Replenishment Problem (JRP), demand is considered to be completely predictable, enabling precise planning and optimization. This often results in longer and more predictable order cycles. Conversely, in a stochastic JRP, demand is uncertain, prompting shorter order cycles to maintain sufficient inventory levels and reduce the likelihood of stockouts, (Braglia & Grazia, 2017; Silver & Hayya, 2006). Accordingly, the deterministic common cycle time is initially used as a starting solution to get the stochastic cycle time.

Stochastic Policy Two (SP2): Uncertain Demand with Production Delay

This policy introduces a deviation from SP1 by permitting planned production delays (y_i) for each item. The retailer's order-up-to level is maintained at the same level $(S_i + SS_i)$ as established in SP1.

Starting with an initial condition at production initiation $(IL(z)_i^p)$ and a defined initial steady-state probability be$\left[\left(IL(z)_i^p = m_i + SS_i\right) = 1\right]$, an iterative approach, resembling the Jacobi method, is used to determine steady-state probabilities and performance measurements

Given that demand is discrete, each unit reduction in m_i corresponds to a production delay y_i equal to $(1/\rho_i)$ units of time. Consequently, $\check{c}_i$ is expressed as $y_i/(k_i T(1-\rho_i))$, if $\check{c}_i \le 1$ only one retailer cycle's inventory will be affected by this delay, as the manufacturer's production in the next cycle$(k_i T(1-\rho_i)p_i)$ in the subsequent cycle is sufficient to restore it to its original level. Otherwise, if $\check{c}_i \ge 1$ then it will require $\check{c}_i$ cycles to fully replenish the inventory, where an amount of $(k_i T(1-\rho_i)p_i)$ is added each cycle until the inventory is restored to its original state.

In order to find the optimum (y_i) for each item an iterative procedure is carried as follows: in each iteration (m_i) is decreased by one unit. Afterward, $\check{c}_i$ is calculated then the steady-state probability for the retailer's inventory level $P\left(IL_i^s(z)\right), P\left(IL_i^e(z)\right)\ \forall \{z = 1\ to\ \check{c}_i\}$ is assessed, the determination of the optimal value (y_i) is achieved through an iterative procedure. In each iteration, the value of (m_i) is decremented by one unit

Subsequently, the steady-state probability for the retailer's inventory level is assessed by shifting the probability distribution accordingly: if $\check{c}_i < 1$ $\left(P\left(IL_i^s(1)\right) \mid m_i = x\right) = k) = \left(P\left(IL_i^s(1)\right) \mid m_i = x-1\right) = k-1)$ and $\left(P\left(IL_i^e(1)\right) \mid m_i = x\right) = k) = \left(P\left(IL_i^e(1)\right) \mid m_i = x-1\right) = k-1)$ if $\check{c}_i \ge 1$ then $\left(P\left(IL_i^s(z)\right) = k + (k_i p_i T)\right) = \left(P\left(IL_i^e(z-1)\right) = k\right)$ where $(k_i p_i T)$ is the maximum production quantity per retailer cycle, while the steady state probabilities for the unaffected cycles remain the same. Hence $E(IL(z))_i^e$ and $E(IL(z))_i^S$ are evaluated and the total system cost becomes:

$$TC_{sp2} = \frac{A+B}{T} + \sum_{i=1}^{n} \Bigg(\frac{\frac{a_i}{k_i} + \frac{b_i}{k_i n_i} + \frac{c_i}{k_i n_i u_i}}{T} + \left[\left(\sum_{z=1}^{\check{c}_i} h_{1i} \left(\frac{E(IL(z))_i^{e+} + E(IL(z))_i^{s+}}{2} \right) + \pi_i \left(\frac{E(IL(z))_i^{e-} + E(IL(z))_i^{s-}}{2} \right) \right) / n_i \right] + \sum_{z=\check{c}_i+1}^{n_i} \left[\left[h_{1i} \left(\frac{E(IL(z))_i^{e+} + E(IL(z))_i^{s+}}{2} \right) + \pi_i \left(\frac{E(IL(z))_i^{e-} + E(IL(z))_i^{s-}}{2} \right) \right] / n_i \right] + h_{2i} \left[\frac{k_i d_i T}{2} \left(n_i - 1 + (2 - n_i)\rho_i\right) + E(o_i)/n_i + \sum_{z=1}^{\check{c}_i} (k_i T(1-\rho_i)) p_i \left(\frac{n_i - z - 1}{n_i} \right) + (\check{c}_i k_i T(1-\rho_i) - y_i) p_i \left(\frac{n_i - z - \check{c}_i}{n_i} \right) + y_i \left(\frac{p_i(n_i - 1)}{n_i} - d_i \right) \right] + h_{3i} \left[\frac{k_i d_i T}{2} (u_i + \rho_i - 1) \right] \Bigg) \tag{12}$$

A search procedure for common cycle time T is carried between the minimum individual cycle time $T_i$and T deterministic from policy 2.

Numerical Analysis:

In this section, we design a numerical experiment to analyze both the deterministic and stochastic cases and demonstrate the cost savings achieved by applying the Joint Replenishment Problem with a Vendor-Managed Inventory (VMI) strategy. This strategy involves optimizing the initiation of production time for each item and the common replenishment time for all orders. We conducted a numerical experiment with an initial population of 10 items and three echelon levels, assuming major and minor costs equal to 300 and shortage cost per unit time of 40 for each item. The following table shows the data used:

Table 1
Data Used for Model Execution

Item	Holding cost Retailer	Holding cost Manufacturer	Order cost Retailer	Setup Cost Manufacture	Demand	Holding cost Supplier	Order Cost Supplier	Production Rate
1	10	7	90	30	15	3	70	70
2	20	5	50	60	30	1	30	50
3	5	3	70	30	20	3	80	70
4	5	2	60	30	10	3	60	12
5	7	4	50	30	30	2	30	50
6	5	3	90	20	10	5	30	90
7	10	8	30	70	15	3	50	70
8	7	2	60	120	20	3	30	25
9	9	7	10	70	70	4	30	110
10	10	7	30	50	20	3	70	30

The application of the first deterministic policy (P1), characterized by the absence of production delays, yields a total system cost of 3597.74 units. The common order cycle for this policy is determined to be 0.861 units of time.

Policy(P2) incorporates production delays at the manufacturer, resulting in increased shortage costs at the retailer while simultaneously decreasing holding costs at both the retailer and the manufacturer. In the first step using the common cycle time from (P1), the optimal delay is evaluated using a bisection search method, dividing the maximum permissible delay within a manufacturer cycle into 100 segments.The subsequent Table 2 presents the cost per item and the multiplier frequency for each item under (P1) and cost reduction employing the production delay.

Table 2
Output Results for P1 Policy

Item	Ki	n_i	u_i	Cost per item (P1)	y_i	c_i	Cost per item (P2)
1	1	1	2	277.79	0	0	277.79
2	1	2	1	448.76	0.4214	2	394.13
3	2	1	1	220.15	0	0	220.15
4	2	2	1	164.30	0.3283	2	160.08
5	1	1	1	264.66	0.0809	1	259.50
6	3	1	1	130.26	0	0	130.26
7	1	2	1	229.07	0	0	229.07
8	1	3	1	270.69	0.2269	2	263.07
9	1	1	1	610.00	0.0895	1	594.62
10	1	2	1	285.36	0.24	1	269.66

An iterative optimization procedure is implemented. Initially, the common cycle time is calculated using Eq. (7). Subsequently, multiplier frequencies are iteratively incremented until an increase in the total system cost is observed. The optimal delay time is determined through a similar iterative process. The determined multiplier frequencies and delay time are then reintroduced into Eq. (7) to recalculate the common cycle time. This iterative process continues until convergence is achieved. Due to the discrete nature of the multiplier frequencies, the system exhibits aperiodic convergence behavior as shown in Fig. 2.

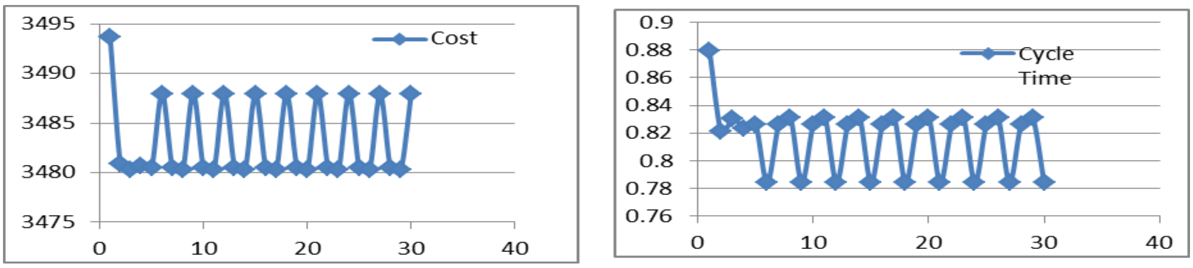

Fig. 2. The cost and cycle time at steady state

The stochastic situation starts with the deterministic solution as the initial condition. An iterative approach is utilized to compute the probabilities of steady-state inventory levels considering the effects of production quantities, backorders, and leftover inventory, and uses Eq. (10) to improve the solution. Table 3 displays the production, backorder, and leftover amounts at the end of the manufacturer cycle for iterations carried until steady state.

Table 3
Iteration Carried Till Reaching Steady State

Iteration	Item 1			Item 2			Item 3			Item 4			Item 5		
	Amount	Back-order	Leftover	Amount	Back-order	Leftover	Amount	Back-order	Leftover	Amount	Back-order	Leftover	Amount	Back-order	Leftover
1	24.13	1.51	0.65	25.43	1.12	1.55	33.77	1.43	1.21	17.92	0.80	1.72	25.43	1.12	1.55
10	26.56	3.36	4.92	26.54	5.19	6.73	35.20	4.83	6.03	18.68	4.21	5.89	26.54	5.19	6.73
20	26.62	3.33	4.94	26.65	5.99	7.64	35.45	5.12	6.57	19.01	4.61	6.62	26.65	5.99	7.64
30	26.62	3.33	4.94	26.77	6.07	7.84	35.46	5.13	6.59	19.64	4.65	7.29	26.77	6.07	7.84
40	26.62	3.33	4.94	26.35	6.42	7.78	35.46	5.13	6.59	19.65	4.67	7.32	26.35	6.42	7.78
50	26.62	3.33	4.94	26.37	6.43	7.80	35.46	5.13	6.59	19.64	4.68	7.32	26.37	6.43	7.80

Iteration	Item 6			Item 7			Item 8			Item 9			Item 10		
	Amount	Back-order	Leftover	Amount	Back-order	Leftover	Amount	Back-order	Leftover	Amount	Back-order	Leftover	Amount	Back-order	Leftover
1	24.04	1.51	0.55	12.64	0.97	0.61	17.64	0.80	1.43	59.72	1.77	2.49	17.38	0.80	1.18
10	27.30	2.46	4.76	13.51	2.13	2.64	18.65	4.10	5.75	60.45	9.03	10.47	18.64	3.78	5.41
20	27.34	2.41	4.76	13.50	2.13	2.63	18.99	4.44	6.43	60.54	11.84	13.38	18.43	4.33	5.76
30	27.34	2.41	4.76	13.50	2.13	2.63	18.57	4.89	6.46	60.62	13.25	14.87	18.45	4.37	5.82
40	27.34	2.41	4.76	13.50	2.13	2.63	18.62	4.87	6.48	60.60	14.10	15.70	18.45	4.38	5.83
50	27.34	2.41	4.76	13.50	2.13	2.63	18.62	4.87	6.49	60.38	14.80	16.18	18.45	4.38	5.83

The safety stock amount required and relevant cost for each cycle is shown in the next table:

Table 4
Safety Stock for Each Item and Backorders

Item	TCr (z=1)	SS(1)	TCr (z>1)	SS(z>1)	$E(IL(1))^{s-}$	$E(IL(1))p^{-}$
1	245.5	8	203.4	4	2.05	9.75E-08
2	495.3	0	356.0	1	1.33	0.031976
3	185.6	16	138.1	7	2.80	2.19E-07
4	141.6	13	79.7	5	0.60	0.515283
5	236.3	13	150.3	6	1.33	0.031976
6	126.9	14	112.1	7	2.53	7.46E-11
7	132.2	6	120.6	3	0.91	0.000175
8	181.9	8	106.0	4	0.68	0.433182
9	585.5	15	368.3	6	2.05	0.0011703
10	223.9	5	142.2	3	0.92	0.163478

2. Conclusion

This study presents a new approach to the Joint Replenishment Planning (JRP) problem by investigating a production-based model with finite capacity, focusing on production initiation timing, backorder management, cycle time, and order frequencies in the presence of both deterministic and stochastic demand, deviating from the common multi-supplier focus .It was found that Use of (VMI) system for a JRP can be helpful for reduction inventory cost through coordination between production and times for shipment. Delaying production at the manufacturer offers the potential to reduce overall supply chain costs by minimizing inventory holding expenses. However, this benefit must be weighed against the risk of increased backorders at the retailer, particularly when backorders exceed the production capacity of a single retailer cycle.

The research investigates a new area of integrating production decisions into joint ordering strategies, which holds significant potential for expansion into remanufacturing. This is particularly applicable when returned items from customers require additional processing on limited-capacity servers before being shipped to the retailer.

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