An EOQ model with time dependent Weibull deterioration and ramp type demand

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**ABSTRACT**

This paper presents an order level inventory system with time dependent Weibull deterioration and ramp type demand rate where production and demand are time dependent. The proposed model of this paper considers economic order quantity under two different cases. The implementation of the proposed model is illustrated using some numerical examples. Sensitivity analysis is performed to show the effect of changes in the parameters on the optimum solution.

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1. Introduction

The control and maintenance of production inventories of deteriorating items with shortages have attracted much attention in inventory analysis. Deterioration plays an important role in developing inventory models since it is a natural process in many cases. Deterioration is normally identified as decay or damage in goods. Foods, drugs, pharmaceuticals, radioactive substances are examples of items in which sufficient deterioration can take place during the normal storage period and thus it plays an important role in analyzing the system.

Shah and Jaiswal (1977), Roychowdhury and Chaudhuri (1983), Dave (1986), Bahari-Kashani (1989), etc studied different types of order-level inventory models for deteriorating items where deterioration rate is considered to be constant. Whitin (1957) considered deterioration of fashion goods at the expiry of prescribed shortage period. Another deteriorating inventory model was developed where deterioration was considered in exponential form (Ghare & Schrader, 1963). Since
then, there have been tremendous works on deteriorating items (Chakrabarti et al., 1998; Covert & Philip, 1973; Mishra, 1975; Goswami & Chaudhuri, 1991, 1992; Fujiwara, 1993; Hariga & Benkherouf, 1994; Wee, 1995; Jalan et al., 1996; Su et al., 1996). To know more work in this line one may consult the review articles written by Nahmias (1982) and Raafat (1991). Traditional inventory problems normally assume that demand is constant and given upfront. However, this simple assumption does not hold in many cases and it can be a function of price, time, etc. Donaldson (1977) is believed to be the first who introduced a linearly time-dependent demand function. There have been tremendous works on time-dependent demand inventory models (McDonald, 1979; Mitra et al., 1984; Ritchie, 1984; Deb & Chaudhuri, 1987; Goyal, 1988; Murdeshwar, 1988; Mandal & Pal, 1998; Panda et al., 2008; Abdul & Murata, 2011). Deng et al. (2007) also presented a review of inventory models for deteriorating items with ramp type demand.

In this paper, we develop economic order quantity (EOQ) models for deteriorating items which are time-dependent and the demand rate is a ramp type function of time. These types of problems are normally observed in the case of new brands of consumer goods. Demand rate for such items usually increases up to certain period and then it almost stabilizes. We assume that the unit production cost and the demand rate to be inversely proportionate. The first model discussed in this paper deals with model where shortage is prohibited and the second one is extended to cover the case of inventory allowing shortage. Two numerical examples are provided to illustrate the solution procedure of our models. Sensitivity analysis is carried out to show the effect of changes in the parameter on the optimum total average cost.

2. Proposed model

2.1. Model 1

To develop the inventory model where shortage is not allowed, the following assumption and notation are used.

a. The lead time is zero.

b. \( c_1 \) is the inventory holding cost per unit per unit of time.

c. \( c_3 \) is the deterioration cost per unit per unit of time.

d. \( R = f(t) \), the demand rate, is assumed to be a ramp type function of time, i.e.

\[
f(t) = D_0 \left[ t - (t - \mu)H(t - \mu) \right], D_0 > 0 , \text{ here } H(t - \mu) \text{ is a Heaviside’s function which may be defined as follows:}
\]

\[
H(t - \mu) = \begin{cases} 1 & \text{if } t \geq \mu, \\ 0 & \text{if } t < \mu. \end{cases}
\]

e. The production rate is \( K = \delta f(t) \) where \( \delta > 1 \) is constant.

f. \( \theta(t) = \alpha \beta t^{\beta-1} \) is the deterioration rate; where \( 0 < \alpha < 1 , t \geq 0 \) and \( \beta > 0 \). Generally \( \alpha \) is called the scale parameter and \( \beta \) is the shape parameter.

g. \( C \) is the total average cost per production cycle.

h. \( t_1 \), the production time is greater than \( \mu \) in no shortage period.
For $\alpha > 0$, $\gamma > 0$ and $\gamma \neq 2$, the unit production cost $v = \alpha R^{-\gamma}$ is positive. Thus $v$ and $R$ are inversely related which implies that higher demands result in lower per unit production costs and $\gamma$ remains positive.

We have,
\[
\frac{dv}{dR} = -\alpha \gamma R^{-(\gamma+1)} < 0, \quad \frac{d^2v}{dR^2} = \alpha \gamma(\gamma + 1)R^{-(\gamma+2)} > 0.
\]

Hence, we observe that the marginal unit cost of production is an increasing function of $R$. Further and with the increase in demand rate, the unit cost of production decreases with an increasing rate resulting encouragement to the manufacturer to produce more as the demand for the item increases. The nature of the solution of the problem requires restriction $\gamma \neq 2$. At initial time $t = 0$, the production starts with zero level stock. At time $t_1$, the production stops as the stock attains $S$ level.

Market demand and deterioration of items gradually diminishes the inventory level during the time period $t_1 \leq t \leq t_2$ which ultimately falls to zero at time $t = t_2$. At time $t = t_2$ the cycle again repeats.

Let $Q(t)$ be the inventory level at any time $t$ ($0 \leq t \leq t_2$).

Differential equations governing the instantaneous states of $Q(t)$ during the time interval $0 \leq t \leq t_2$ are as follows,
\[
\frac{dQ(t)}{dt} + \theta(t)Q(t) = K - f(t), \quad 0 \leq t \leq \mu \tag{1}
\]
\[
\frac{dQ(t)}{dt} + \theta(t)Q(t) = K - f(t), \quad \mu \leq t \leq t_1 \tag{2}
\]
\[
\frac{dQ(t)}{dt} + \theta(t)Q(t) = -f(t), \quad t_1 \leq t \leq t_2 \tag{3}
\]
satisfying the conditions $Q(t_1) = S$, $Q(t_2) = 0$.

Using $\theta(t) = \alpha \beta t^{\beta-1}$ and ramp type function $f(t)$, Eq. (1) to Eq. (3) we have the following,
\[
\frac{dQ(t)}{dt} + \alpha \beta t^{\beta-1} Q(t) = (\delta - 1)D_0 t, \quad 0 \leq t \leq \mu \tag{4}
\]
satisfying the initial condition $Q(0) = 0$,
\[
\frac{dQ(t)}{dt} + \alpha \beta t^{\beta-1} Q(t) = (\delta - 1)D_0 \mu, \quad \mu \leq t \leq t_1 \tag{5}
\]
satisfying the condition $Q(t_1) = S$. 

\[
\frac{dQ(t)}{dt} + \alpha \beta t^{\beta-1} Q(t) = -D_0 \mu, \quad t_1 \leq t \leq t_2
\]

satisfying the conditions \(Q(t_1) = S\), \(Q(t_2) = 0\).

Solving the Eqs. (4)-(6) yields,

\[
Q(t) = \begin{cases} 
(\delta - 1)D_0 \left( \frac{t^2}{2} + \frac{\alpha \beta + 2}{\beta + 2} - \frac{\alpha \beta + 2}{2} \right) & \text{if } 0 \leq t \leq \mu \\
(\delta - 1)D_0 \left[ t + \frac{\alpha \mu^{\beta+1}}{\beta + 1} - \alpha \beta + 2 - \frac{\mu}{\beta + 2} - \frac{\alpha \mu^{\beta+1}}{\beta + 1} - \frac{\alpha \mu t^{\beta+1}}{2} \right] & \text{if } \mu \leq t \leq t_1 \\
S(1 - \alpha \beta + \alpha t_1^{\beta}) + D_0 \mu \left[ t_1 - t + \frac{\alpha}{\beta + 1} (t_1^{\beta+1} - t_2^{\beta+1}) + \alpha \beta (t - t_1) \right] & \text{if } t_1 \leq t \leq t_2
\end{cases}
\]

We neglect the second and higher powers of \(\alpha\) throughout the subsequent calculations as \(0 < \alpha < 1\). Since \(Q(t_2) = 0\), from Eq. (7), we get,

\[
S(1 - \alpha t_2^{\beta} + \alpha t_1^{\beta}) + D_0 \mu \left[ t_2 - t_1 + \frac{\alpha}{\beta + 1} (t_2^{\beta+1} - t_1^{\beta+1}) + \alpha \beta (t_1 - t_2) + \alpha t_2^{\beta+1} - \alpha t_2^{\beta} t_1 - \alpha t_1^{\beta} t_2 + \alpha t_1^{\beta+1} \right] = 0.
\]

Simplifying and taking the first order approximation over \(\alpha\) yields,

\[
S = D_0 \mu \left[ t_2 - t_1 + \frac{\alpha}{\beta + 1} (t_2^{\beta+1} - t_1^{\beta+1}) + \alpha \beta (t_1 - t_2) + \alpha t_2^{\beta+1} - \alpha t_2^{\beta} t_1 - \alpha t_1^{\beta} t_2 + \alpha t_1^{\beta+1} \right].
\]

The total inventory in \(0 \leq t \leq t_2\) is as follows,

\[
\int_0^\mu Q(t) dt + \int_{t_1}^{t_2} Q(t) dt = (\delta - 1)D_0 \left[ \frac{\mu^3}{6} + \frac{\alpha \mu^{\beta+3}}{(\beta + 2)(\beta + 3)} - \frac{\alpha \mu^{\beta+3}}{2(\beta + 3)} \right] + (\delta - 1)D_0 \mu \left[ \frac{t_1^2}{2} + \frac{\alpha t_1^{\beta+2}}{(\beta + 1)(\beta + 2)} - \frac{\alpha t_1^{\beta+2}}{\beta + 2} - \frac{\mu t_1^{\beta+2}}{2} \right]
\]

\[
- \frac{\alpha \mu^{\beta+1} t_1}{\beta + 2} - \frac{\alpha \mu t_1^{\beta+1}}{\beta + 2} - \frac{\alpha \mu t_1^{\beta+1}}{2(\beta + 1)} - \frac{\alpha t_1^{\beta+2}}{(\beta + 1)(\beta + 2)} + \frac{\alpha t_1^{\beta+2}}{\beta + 2} + \frac{2 \alpha t_1^{\beta+2}}{\beta + 2}
\]

\[
- \frac{\alpha t_2^{\beta+2}}{(\beta + 1)(\beta + 2)} + \frac{\alpha t_2^{\beta+2}}{\beta + 2} - \frac{\alpha t_2^{\beta+2}}{\beta + 1} + \frac{\alpha t_1^{\beta+2}}{\beta + 1} + \frac{t_2^2}{2} + \frac{\alpha t_1^{\beta+2}}{\beta + 1} + \frac{\alpha t_1^{\beta+2}}{\beta + 2} \right]
\]

The total number of deteriorated items in \(0 \leq t \leq t_2\) is given by

\[
\delta \int_0^\mu D_0 \mu dt + \delta \int_{t_1}^{t_2} D_0 \mu dt - \int_0^\mu D_0 \mu dt - \int_{t_1}^{t_2} D_0 \mu dt = \frac{1}{2} D_0 \delta \mu (2t_1 - \mu) - \frac{1}{2} D_0 \mu (2t_2 - \mu).
\]

The cost of production in \([u, u + du]\) is \(Kvdu = \frac{\alpha \delta}{R^{\gamma-1}} du\). So the production cost in \(0 \leq t \leq t_1\) is
\[
\int_0^t \frac{\alpha_1 \delta}{R^2} du + \int_0^t \frac{\alpha_2 \delta}{R^2} du = \frac{\alpha_1 \delta D_0^{-1/2}}{2 - \gamma} (2 - \gamma)^{1/2} t_1 + (\gamma - 1)^{1/2} t_2, \quad \gamma \neq 2
\] (11)

Thus the total average cost is as follows,

\[
C = \frac{1}{t_2} \left[ (\delta - 1) D_0 c_1 \left\{ \frac{\mu^2}{6} + \frac{\alpha_1 \mu^{\beta+3}}{2(\beta+3)} - \frac{\alpha_2 \mu^{\beta+3}}{2(\beta+3)} \right\} + (\delta - 1) c_1 D_0 \mu \left\{ \frac{t_1^2}{2} + \frac{\alpha_1 t_1^{\beta+2}}{(\beta+1)(\beta+2)} - \frac{\alpha_1 t_1^{\beta+2}}{\beta+1} \right\} \right] + c_1 D_0 \mu \left\{ \frac{t_2^2}{2} - \frac{\alpha_2 t_2^{\beta+2}}{2(\beta+1)} + \frac{\alpha_2 t_2^{\beta+2}}{\beta+2} \right\} + \frac{1}{2} c_2 D_0 \mu (2t_1 - \mu) - \frac{1}{2} c_2 D_0 \mu (2t_2 - \mu) + \frac{\alpha_0 \delta D_0^{-1/2}}{2 - \gamma} \left\{ (2 - \gamma)^{1/2} t_1 + (\gamma - 1)^{1/2} t_2 \right\}
\] (12)

We can find the optimum values of \( t_1 \) and \( t_2 \) for minimum average cost \( C \) from the solutions of the following equations

\[
\frac{\partial C}{\partial t_1} = 0 \quad \text{and} \quad \frac{\partial C}{\partial t_2} = 0,
\] (13)

where

\[
\frac{\partial^2 C}{\partial t_1^2} > 0, \quad \frac{\partial^2 C}{\partial t_2^2} > 0, \quad \text{and} \quad \frac{\partial^2 C}{\partial t_1 \partial t_2} > 0.
\]

From Eq. (13) we get

\[
c_1 (\delta - 1) D_0 \mu \left\{ t_1 + \frac{\alpha_1 t_1^{\beta+1}}{2(\beta+1)} + \alpha_1 t_2^{\beta+1} + \frac{\alpha_1 t_2^{\beta+1}}{2(\beta+1)} \right\} + c_1 D_0 \mu \left\{ \frac{t_1^2}{2} + \frac{\alpha_1 t_1^{\beta+2}}{(\beta+1)(\beta+2)} - \frac{\alpha_1 t_1^{\beta+2}}{\beta+1} \right\} + c_1 D_0 \mu \left\{ \frac{t_2^2}{2} - \frac{\alpha_2 t_2^{\beta+2}}{2(\beta+1)} + \frac{\alpha_2 t_2^{\beta+2}}{\beta+2} \right\} + c_1 D_0 \mu \left\{ \frac{t_2^2}{2} - \frac{\alpha_2 t_2^{\beta+2}}{2(\beta+1)} + \frac{\alpha_2 t_2^{\beta+2}}{\beta+2} \right\} + \frac{1}{2} c_2 D_0 \mu (2t_1 - \mu) - \frac{1}{2} c_2 D_0 \mu (2t_2 - \mu) + \frac{\alpha_0 \delta D_0^{-1/2}}{2 - \gamma} \left\{ (2 - \gamma)^{1/2} t_1 + (\gamma - 1)^{1/2} t_2 \right\} = 0
\] (14)

\[
\frac{1}{t_2} \left[ (\delta - 1) D_0 c_1 \left\{ \frac{\mu^3}{2(\beta+3)} - \frac{\alpha_1 \mu^{\beta+3}}{2(\beta+3)} \right\} + (\delta - 1) c_1 D_0 \mu \left\{ \frac{t_1^2}{2} + \frac{\alpha_1 t_1^{\beta+2}}{(\beta+1)(\beta+2)} - \frac{\alpha_1 t_1^{\beta+2}}{\beta+1} \right\} \right] + c_1 D_0 \mu \left\{ \frac{t_1^2}{2} - \frac{\alpha_1 t_1^{\beta+2}}{2(\beta+1)} + \frac{\alpha_1 t_1^{\beta+2}}{\beta+2} \right\} + c_2 D_0 \mu \left\{ (2t_1 - \mu) + \frac{1}{2} c_2 D_0 \mu (2t_2 - \mu) + \frac{\alpha_0 \delta D_0^{-1/2}}{2 - \gamma} \left\{ (2 - \gamma)^{1/2} t_1 + (\gamma - 1)^{1/2} t_2 \right\} \right\} = 0
\] (15)
2.2. Model 2

In this section, we develop a model for deteriorating items when shortage is permitted and completely backlogged and a finite rate of replenishment is assumed for planning horizon. Let $c_2$ be the shortage cost per unit per unit of time. We start with zero stock at the initial stage. At $t = 0$, production starts and continues till $t = t_1$. At this time the stock reaches $S$ level and production is stopped at $t = t_1$. Accumulated inventory during $0 \leq t \leq t_1$, after meeting the demands during $0 \leq t \leq t_1$, is available for meeting the demand during $t_1 \leq t \leq t_2$. The stock is exhausted or reaches zero level at time $t_2$. Once demand is not satisfied, shortages start to develop and accumulates up to the level $P$ at $t = t_3$. Again, after time $t_3$, production starts and inventory reaches zero level at time $t_4$ satisfying the demand during the period $t_3 \leq t \leq t_4$ along with the backlogged shortages during the period $t_2 \leq t \leq t_3$. At $t = t_4$, the production cycle completes and new cycle starts. The purpose of the study is to determine the optimum values of $C$, $t_1$, $t_2$, $t_3$ and $t_4$ subject to the assumptions stated earlier.

Let $Q(t)$ be the inventory level at any time $t$ ($0 \leq t \leq t_4$). The flowing differential equations represent the instantaneous states of $Q(t)$ during the time interval $0 \leq t \leq t_4$.

\[
\frac{dQ(t)}{dt} + \alpha \beta t^{\beta-1} Q(t) = (\delta - 1)D_0t, \quad 0 \leq t \leq \mu
\]

satisfying the initial condition $Q(0) = 0$,

\[
\frac{dQ(t)}{dt} + \alpha \beta t^{\beta-1} Q(t) = (\delta - 1)D_0\mu, \quad \mu \leq t \leq t_1
\]

satisfying the condition $Q(t_1) = S$,

\[
\frac{dQ(t)}{dt} + \alpha \beta t^{\beta-1} Q(t) = -D_0\mu, \quad t_1 \leq t \leq t_2
\]

satisfying the conditions $Q(t_1) = S$, $Q(t_2) = 0$,

\[
\frac{dQ(t)}{dt} = -D_0\mu, \quad t_2 \leq t \leq t_3
\]

satisfying the conditions $Q(t_2) = 0$, $Q(t_3) = -P$,

\[
\frac{dQ(t)}{dt} = (\delta - 1)D_0\mu, \quad t_3 \leq t \leq t_4
\]

satisfying the conditions $Q(t_3) = -P$, $Q(t_4) = 0$.

From Eq. (7), the solutions of the Eq. (16) to Eq. (18) can be obtained and the solutions of Eq. (19) and Eq. (20) are as follows,
\[
\begin{cases}
(\delta - 1)D_0\left(\frac{t^2}{2} + \frac{\alpha t^{\beta + 2}}{\beta + 2} - \frac{\alpha t^{\beta + 2}}{2}\right) & \text{if } 0 \leq t \leq \mu \\
(\delta - 1)D_0\mu \left(t + \frac{\alpha t^{\beta + 1}}{\beta + 1} - \frac{\mu}{\beta + 2} - \frac{\alpha \mu t^{\beta + 1}}{\beta + 1} - \frac{\alpha \mu t^{\beta + 1}}{2}\right) & \text{if } \mu \leq t \leq t_1 \\
Q(t) = S(1 - \alpha t^\beta + \alpha t_1^\beta) + D_0\mu \left(t - t_1 - \frac{\alpha}{\beta + 1}(t_1^{\beta + 1} - t^{\beta + 1}) + \alpha t^\beta (t - t_1)\right) & \text{if } t_1 \leq t \leq t_2 \\
-D_0\mu(t - t_2) & \text{if } t_2 \leq t \leq t_3 \\
(\delta - 1)D_0\mu(t - t_4) & \text{if } t_3 \leq t \leq t_4
\end{cases}
\]

During the time \( t_2 \leq t \leq t_4 \), there is no deterioration as the items produced are sent for meeting the demand, immediately. Hence, total number of deteriorated items during the time \( 0 \leq t \leq t_4 \) will be the same as the one given in Eq. (10) i.e.

\[
\frac{1}{2}D_0\delta\mu(2t_1 - \mu) - \frac{1}{2}D_0\mu(2t_2 - \mu).
\]

The total shortage during the time \( t_2 \leq t \leq t_4 \) is as follows,

\[
\int_{t_2}^{t_4} [-Q(t)]dt + \frac{1}{2}D_0\mu(t_3 - t_2)^2 + \frac{1}{2}(\delta - 1)D_0\mu(t_4 - t_3)^2,
\]

and production cost during the time \( t_3 \leq t \leq t_4 \) is as follows,

\[
\int_{t_3}^{t_4} K\nu\alpha t = \alpha t_1^\gamma D_0^{1-\gamma} \mu^{1-\gamma} (t_4 - t_3).
\]

Hence the cost of production during the time \( 0 \leq t \leq t_4 \) is computed as,

\[
\alpha t_1^\gamma D_0^{1-\gamma} - \left[(2 - \gamma)\mu^{1-\gamma} (t_4 + t_4 - t_3) + (\gamma - 1)\mu^{2-\gamma}\right]. \quad \gamma \neq 2
\]

The total average cost of the system during the time \( 0 \leq t \leq t_4 \) is as follows,

\[
C = \frac{1}{t_4} \left[ (\delta - 1)D_0c_1 \left(\frac{\mu^3}{6} + \frac{\alpha \mu^{\beta + 3}}{\beta + 3} - \frac{\alpha \mu^{\beta + 3}}{2(\beta + 3)}\right) + (\delta - 1)c_1D_0\mu \left(\frac{t_4^2}{2} + \frac{\alpha t_1^{\beta + 2}}{\beta + 2} - \frac{\alpha t_1^{\beta + 2}}{\beta + 2}\right) \right]
\]

\[
+ \alpha t_2^{\beta + 2} \left(\frac{\alpha t_2^{\beta + 2}}{\beta + 2} - \frac{\alpha t_1^{\beta + 2}}{\beta + 2}\right) + \alpha t_3^{\beta + 2} \left(\frac{\alpha t_3^{\beta + 2}}{\beta + 2} - \frac{\alpha t_1^{\beta + 2}}{\beta + 2}\right) + \frac{1}{2}D_0\mu c_2(t_3 - t_2)^2
\]

\[
+ \frac{1}{2}D_0\mu c_2(\delta - 1)(t_4 - t_3)^2 + \frac{1}{2}c_3D_0\delta\mu(2t_1 - \mu) - \frac{1}{2}c_3D_0\mu(2t_2 - \mu)
\]

\[
+ \frac{1}{2}D_0t_1^\gamma \mu^{1-\gamma} (t_4 + t_4 - t_3) + (\gamma - 1)\mu^{2-\gamma}\right]
\]

\[
, \quad \gamma \neq 2
\]
The required optimum values of \( t_1, t_2, t_3 \) and \( t_4 \) which minimize the cost function \( C \) can be obtained from the solution of the following equations,

\[
\frac{\partial C}{\partial t_1} = 0, \quad \frac{\partial C}{\partial t_2} = 0, \quad \frac{\partial C}{\partial t_3} = 0 \quad \text{and} \quad \frac{\partial C}{\partial t_4} = 0,
\]

subject to the conditions that these values of \( t_j(i=1,2,3,4) \) satisfy the conditions \( D_i > 0(i=1,2,3,4) \), where \( D_i \) is the Hessian determinant of order \( i \) given by

\[
D_i = \begin{bmatrix}
    c_{i1} & c_{i2} & \ldots & c_{ii} \\
    c_{2i} & c_{22} & \ldots & c_{2i} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{ii} & c_{i2} & \ldots & c_{ii}
\end{bmatrix},
\]

\[
c_{ij} = \frac{\partial^2 C}{\partial t_i \partial t_j} \quad (i, j = 1, 2, 3, 4)
\]

From Eq. (23) we get,

\[
c_1 \left\{ (\delta - 1)D_0 \mu \left( t_1 + \frac{\alpha t_1}{\beta + 1} - \alpha t_1 - \frac{\mu}{2} - \frac{\alpha}{\beta + 2} + \frac{\alpha}{\beta + 1} - \frac{\alpha}{\beta + 2} \right) + D_0 \mu \left( \frac{\alpha t_1}{\beta + 1} + \frac{\alpha}{\beta + 1} \right) \right\} + \alpha_i \delta D_0^{i-\gamma} \mu^{i-\gamma} = 0,
\]

\[
c_1 D_0 \mu \left( 2t_2 - t_2 - \frac{\alpha t_2}{\beta + 1} + \alpha t_2 - t_1 + \frac{\alpha}{\beta + 1} \right) - c_3 D_0 \mu - c_2 D_0 \mu (t_3 - t_2) = 0
\]

\[
D_0 \mu c_3 (t_3 - t_2) - c_2 (\delta - 1)D_0 \mu (t_4 - t_3) - \alpha_i \delta D_0^{i-\gamma} \mu^{i-\gamma} = 0,
\]

\[
\frac{1}{t_4} \left\{ (\delta - 1)D_0 \mu \left( \frac{\mu}{6} + \frac{\alpha}{\beta + 1} - \frac{\alpha}{\beta + 3} \right) + (\delta - 1)c_1 D_0 \mu \left( \frac{t_1^2}{2} + \frac{\alpha t_1}{\beta + 1} + \frac{\alpha}{\beta + 1} \right) - \frac{\mu}{2} \right\} + \frac{\alpha}{\beta + 1} \left( \frac{2\alpha}{(\beta + 1)(\beta + 2)} + \frac{\alpha}{\beta + 1} + \frac{\alpha}{\beta + 1} \right) + \frac{\alpha}{\beta + 1} \left( \frac{2\alpha}{(\beta + 1)(\beta + 2)} + \frac{\alpha}{\beta + 1} + \frac{\alpha}{\beta + 1} \right) + c_1 D_0 \mu \left( \frac{t_2^2}{2} + \frac{\alpha t_2}{(\beta + 1)(\beta + 2)} + \frac{\alpha}{\beta + 1} \right)
\]

\[
+ \frac{1}{2} D_0 \mu c_2 (t_3 - t_2)^2 + D_0 \mu c_3 (\delta - 1)t_3^2 + \frac{1}{2} c_1 D_0 \delta \mu (2t_4 - \mu) - \frac{1}{2} c_3 D_0 \mu (2t_2 - \mu)
\]

\[
+ \alpha_i \delta D_0^{i-\gamma} \left\{ (2 - \gamma)\mu^{i-\gamma} (t_1 - t_1) + (\gamma - 1)\mu^{2-\gamma} \right\} \right\} + c_2 D_0 \mu (\delta - 1)(t_4 - t_1) + \alpha_i \delta D_0^{i-\gamma} \mu^{i-\gamma} = 0.
\]

3. Numerical Example

Example 1. Consider \( c_1 = 4, c_3 = 10, D_0 = 100, \mu = 12, \alpha = 0.005, \beta = 0.4, \delta = 8, \alpha_i = 18 \) and \( \gamma = 1.2 \) as appropriate units. Using the Mathematica-5.1, we obtain the optimum solution for \( t_i \) and
Equation (14) and Eq. (15) of Model 1, as \( t_1^* = 3.90269 \), \( t_2^* = 125.476 \). Using \( t_1^* \) and \( t_2^* \) in Eq. (12), we get the optimum average cost as \( C^* = 48534.5 \).

Example 2. Consider \( c_1 = 4 \), \( c_2 = 6 \), \( c_3 = 10 \), \( D_0 = 100 \), \( \mu = 12 \), \( \alpha = 0.005 \), \( \beta = 0.4 \), \( \delta = 8 \), and \( \gamma = 1.2 \) as appropriate units. Using the Mathematica-5.1, we obtain the optimum solution for \( t_1^* \), \( t_2^* \), \( t_3^* \) and \( t_4^* \) of Eq. (24) to Eq. (27) of Model 2 as \( t_1^* = 3.61333 \), \( t_2^* = 32.5721 \), \( t_3^* = 48.7701 \) and \( t_4^* = 51.0835 \). Using \( t_1^* \), \( t_2^* \), \( t_3^* \) and \( t_4^* \) in Eq. (22), we get the optimum average cost as \( C^* = 230578 \).

4. Sensitivity Analysis

We have performed sensitivity analysis by changing one parameter at a time by 25% and 50%, and keeping the remaining parameters at their original values. Table 1 and Table 2 summarize the results.

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<th>( C^* )</th>
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Based on the results of Table 1, the following observations can be made.
(i) An increase on the values of the parameters $c_1$, $c_3$, $D_0$, $\mu$, $\alpha$, $\beta$, $\delta$ and $\alpha_1$ will result to an increase on $C^*$.

(ii) An increase in the values of the parameter $\gamma$ will result to an in decrease on $C^*$.

Table 2
The summary of the results when shortage is permitted

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Based on the results of Table 2, the following observations can be made.
4.2.1. Any increase in the values of the parameters $c_1$ and $\mu$ will result to an increase on $C^*$. 

4.2.2. Any increase in the values of the parameters $c_2$, $c_3$, $D_0$, $\alpha$ and $\delta$ will result to an increase on the value of $C^*$. 

4.2.3. Any increase in the values of the parameters $\beta$ will result to a decrease in $C^*$. 

4.2.4. Any increase in the values of the parameters $\alpha_1$ and $\gamma$ will result in slight change in $C^*$. 

As we can observe from the results of Table 1 and Table 2, the optimal average cost obtained in no shortage case is more than that of shortage case. 

5. Conclusion

In this paper, we have presented a new economic order quantity model when the demand rate is a ramp type function of time. The ramp type demand is generally observed in new brand of consumer goods where demand increases for a certain period and then it stabilizes and becomes almost constant. The proposed model of this paper is considered for two different conditions where shortage is either prohibited for the first case and it is permitted for the second one. The proposed model was analyzed using two numerical examples and they were analyzed when parameters are set to different values.

Acknowledgment

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References


