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International Journal of Industrial Engineering Computations

homepage: www.GrowingScience.com/ijiec

MILP model for simultaneous batching, production and distribution operations in single-stage multiproduct batch plants

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C H R O N I C L E

Article history:
Received January 12 2025
Received in Revised Format March 8 2025
Accepted April 8 2025
Available online April 8 2025

Keywords:
Production and distribution
Short-term
Batch environment
MILP
Integrated approach

A B S T R A C T

Traditionally, the short-term production and distribution activities have been addressed with a decoupled and sequential methodology. Although this approach simplifies the problem, there are several environments where it generates inefficiencies or is simply not applicable. Consequently, the integration of both problems is very valuable in a variety of industrial applications, especially in industries where final products must be delivered to customers shortly after production. This paper presents a mixed-integer linear optimization model that simultaneously solves the production and distribution scheduling in a single-stage multi-product batch facility with multiple non-identical units operating in parallel, where transportation operations are carried out with a heterogeneous fleet of vehicles. As operations are performed in a batch environment, the production and distribution problems also integrate decisions related to the number and size of batches required to meet the demand for multiple products. The capabilities of the proposed approach are illustrated through several cases of study. Finally, these examples are solved with a two-stage approach and the superiority of the solutions using the integrated approach is demonstrated.

Nomenclature

Indices

b	batch
d	time window
i	customer
i_0	plant
l	slot
p	product
u	processing unit
v	vehicle
vt	type of vehicle

Sets

B_p	set of batches proposed for product p
D	set of time windows
IC	set of customers
L_u	set of slots proposed for unit u
P	set of products
U	set of units
V	set of vehicles
VT	set of vehicle types

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ISSN 1923-2934 (Online) - ISSN 1923-2926 (Print)
2025 Growing Science Ltd.
doi: 10.5267/j.ijiec.2025.4.005

VVT_{vt}	subset of vehicles of type *vt*

Parameters

a_d	lower bound for time window *d*
cap_{vt}	capacity of a vehicle of type *vt*
$capmax_{up}$	maximum production capacity of unit *u* for processing product *p*
$capmin_{up}$	minimum production capacity of unit *u* for processing product *p*
c_d	upper bound for time window *d*
dem_{ipd}	amount of demand of product *p* from customer *i* to be delivered in time window *d*
$dist_{ii'}$	distance between nodes *i* and *i'*
fpc_{up}	processing cost of a batch of product *p* in unit *u*
fpt_{up}	processing time of a batch of product *p* in unit *u*
ftc_{vt}	fixed transport cost for vehicles of type *vt*
M_n	parameter used for constraint of Big-M type, where n =1, 2, 3, 4
MTD	shortest travel time between the plant and a customer
NB^{UP}_{up}	maximum number of batches of product *p* that can be processed in unit *u*
nb_p	maximum number of batches of the product *p*
ns_u	maximum number of slots of the unit *u*
prc_{vt}	occupancy minimum rate for vehicles of type *vt*
$tv_{ii'}$	travel time between nodes *i* and *i'* for vehicles
vtc_{vt}	variable transport cost per unit of traveled distance for vehicles of type *vt*
α_p	size factor representing the weight per unit of final product *p*

Binary Variables

R_{bpv}	indicates if batch *b* of product *p* is assigned to vehicle *v*
X_{bpul}	indicates if batch *b* of product *p* is processed in unit *u* in slot *l*
$Y_{ii'v}$	indicates if customer *i* is visited immediately before customer *i'* with vehicle *v*
W_v	indicates if vehicle *v* is used
Z_{idv}	indicates if the order *d* of customer *i* is delivered with the vehicle *v*
ZP_{iv}	indicates if customer *i* is the first on the route of vehicle *v*
ZU_{iv}	indicates if customer *i* is the last on the route of vehicle *v*

Integer Variables

SP_{up}	number of feasible slots that can be allocated in unit u of product *p*

Continuous Variables

BS_{bp}	size of batch *b* of product *p*
DET_{iv}	arrival time at customer *i* in vehicle *v*
DT_v	departure time of vehicle *v*
FT_{ul}	final time of slot *l* processing in unit *u*
QT_{bpv}	quantity of batch *b* of product *p* loaded in the vehicle *v*
ST_{ul}	start time of slot *l* processing in unit *u*

1. Introduction

Production and distribution are two closely interrelated activities, mainly because the transportation of final products can only begin after all tasks in the production process have been completed. Even in companies that decouple them through inventory, the integrated management of these operations is a key tool for achieving greater efficiency in the operations of the company. Moreover, this integration is extremely valuable in the presence of highly variable markets, which require more attention in the manufacturing of customized products, in supply chains with time-sensitive products that have a very limited shelf lifespan (Atasagun & Karaoğlan, 2024) such as home chemotherapy delivery (Arda et al., 2024), dairy products (Guarnaschelli et al., 2020) and ready-mixed concrete (Yin et al., 2023; Tibaldo et al., 2025), in make-to-order production systems (MTO), or in those that implement a just-in-time (JIT) policy (Hein and Almeder, 2016), where very little or no inventory of finished products is required. In these environments, production and delivery operations must be accurately synchronized and jointly scheduled, so that final products are shipped to customers shortly after production in order to respond quickly to their needs and improve overall system performance and optimize some established measures.

At the operational level, both production and distribution problems have been extensively studied individually in the area of Process System Engineering (PSE), applying quantitative techniques for resource optimization and decision making in the field of Operations Research (OR). In particular, in batch processes industries, characterized by their flexibility and ability to produce a wide variety of products sharing the same resources, the short-term production scheduling problem is of remarkable importance. In general terms, this problem consists of the following decisions: (a) selection and sizing of batches to be processed (batching), (b) assignment of batches to processing units, (c) sequencing of batches on units, and (d) timing of batches. Excellent reviews on this issue, addressing different modeling approaches and solution methods can be found in Méndez et al. (2006), Maravelias (2012), Harjunkoski et al. (2014), and Castro et al. (2018). However, few works have included batching decisions in the production scheduling problem, mainly due to the combinatorial nature of the involved decisions (Özbel & Baykasoglu, 2023). Thus, through a holistic approach, cost reduction and better use of available resources, among others, may be achieved. A recent study, presented by Ackermann et al. (2021), shows that the integration of these problems may be even more advantageous if all orders of a specific product are consolidated into a single demand for that product, instead of dealing with the batching and scheduling of each order separately.

Likewise, numerous approaches with different assumptions and solution algorithms have been proposed in the literature to address the transportation problem, usually referred to as the classical vehicle routing problem (VRP). Formally, the typical VRP consists of determining the optimal delivery routes to serve a set of customers, geographically dispersed around a central depot, such that each customer location is visited by only one vehicle, each vehicle starts and ends its route at the depot, and the total demand of the customers served by a route does not exceed the assigned vehicle capacity. A broad range of problem variants and extensions based on customer-related, vehicle-related and depot-related aspects, that allow the incorporation of different real-life features or scenarios, have been tackled in comprehensive reviews on this topic (Braekers et al., 2016; Tan & Yeh, 2021; Toth & Vigo, 2014). Although these characteristics make the models more realistic and their solutions more applicable in practice, they bring along a significant level of complexity that requires the development of efficient methodologies to solve these variants.

Traditionally, the production and transportation scheduling problems have been solved separately and sequentially (Berghman et al., 2023; Ceylan et al., 2019; Chen, 2004; Kumar et al., 2020; Moons et al., 2017). In few cases, the batching problem is addressed first, where the number of batches of each product to be processed to meet demand, as well as their size, is determined. The obtained batches, or those proposed by the user if the batching problem is not solved, are used as inputs into the production scheduling model, which is solved to determine where, when and how each batch is processed in the processing units. Based on these decisions, the distribution stage is carried out, which involves decisions regarding the number and type of vehicles to be used, assignment of orders to vehicles, sequencing of shipments, vehicle dispatch times and arrival at customers, and used routes. This methodology, based on optimizing each problem independently, ignores the requirements and constraints of the other, which often can lead to suboptimal solutions as well as not satisfy the customer's expectations. The studies of Moons et al. (2017), Yağmur and Kesen (2023), and more recently Berghman et al. (2023), on the coordination of such problems at the operational decision level, point out that Integrated Production and Distribution Scheduling Problem (IPDSP) can achieve economic savings between 5% and 20% compared to sequential decision making. However, from the mathematical point of view, taking into account that these problems are highly combinatorial, the development of representations that integrate all decisions can lead to computationally expensive and intractable models. Thus, the simultaneous resolution of these activities is a great challenge.

Since the past decade, an increased number of research attempts on IPDSP models, as well as some particular approaches motivated by different practical applications, have been developed in the area literature. The main articles that review the existing works on the operational IPDSP problem propose a classification scheme based on different characteristics: production environment, delivery and routing aspects, fleet type, objective function, and solution approaches (Berghman et al., 2023; Ceylan et al., 2019; Moons et al., 2017). Most studies that address the integration of these decisions analyze the problem in relatively simple production environments, where orders must be processed in a single stage with a single unit (Devapriya et al., 2017; Ganji et al., 2022; Miranda et al., 2019) or with several units in parallel (Jiang et al., 2020; Kesen & Bektas, 2019). In terms of delivery operations, approaches in which vehicles visit a set of customers in the same route (Ullrich, 2013; Yağmur & Kesen, 2023) are predominant, highlighting the importance of allocation, sequencing and timing decisions. However, some studies consider less complex scenarios, where each vehicle delivers directly to a single customer (Eray Cakici & Kurz, 2012; Noroozi et al., 2018). Most contributions focus on economic performance measures (Belo-Filho et al., 2015; Lee et al., 2014), which do not always accurately reflect the objectives of different stakeholders. Moreover, due to the inherent complexity of these problems, approximate approaches, such as decomposition algorithms and heuristics (Jamili et al., 2016; Liu et al., 2021), are more common than exact methodologies (Karaoğlan & Kesen, 2017; Zu et al., 2014), as they allow obtaining suboptimal solutions in reasonable computational times.

Despite the aforementioned works, there are very few studies that integrate batching decisions into IPDSP in the way they have been approached in this paper. Most researches that simultaneously address batching decisions and IPDSP define a batch as a set of orders from different customers, grouped to be processed together and delivered on a single route using a specific vehicle, i.e., it is not allowed to split an order into several batches to be processed in different units (Devapriya et al., 2017; Farmand et al., 2021). The concept of dividing orders into several batches for processing was first introduced and analyzed in the context of the IPDSP problem by Amorim et al. (2013). The authors demonstrated that including batching decisions can generate more efficient solutions by explicitly considering the perishability of the products. In more general contexts, this option adds flexibility to the production system, improves delivery times and optimizes the use of production capacity and plant resources. Another of the few works on the subject is the one presented by Cóccola et al. (2013). The authors propose a MILP model that integrates batching, production scheduling and distribution in an environment with multiple plants and non-identical units in parallel. Although all decisions are studied in an integrated manner, batches can be assigned directly to vehicles or sent to distribution centers, which provides greater flexibility and allows decoupling the coordination of production and distribution activities. The model also allows the incorporation of time constraints such as delivery dates, but these are not described in detail and effectively solved. The above studies reveal that research on the integration of batching decisions in the context of short-term IPDSP is really scarce, which underlines the importance of continuing to explore this field in order to broaden and deepen its understanding.

In this context, the present paper proposes a MILP model that simultaneously solves the batching, production and distribution scheduling problems for the case of a single-stage multi-product batch facility with multiple non-identical units operating in parallel, where a heterogeneous fleet of vehicles of different capacities and costs performs transportation operations. One of the strongest assumptions of the paper is that it is necessary to satisfy the total number of orders issued by customers for each pre-established time window while minimizing total cost. An efficient formulation is presented that allows reaching the optimal solution of the integrated problem in large examples in reasonable computation times.

The rest of the article is organized as follows. Section 2 describes the problem and introduces the notation used in the mathematical formulation of the model. In Section 3, the problem formulation is presented, showing the variables and constraints used. Section 4 includes three examples that demonstrate the efficiency and effectiveness of the proposed approach. Furthermore, in Section 5, to highlight the importance of the integrated solution of these problems, the examples presented in Section 4 are addressed using a sequential approach, and the solutions obtained are compared with those generated by the integrated approach. Finally, in Section 6, the conclusions of the article are presented.

2. Problem description

The problem considered in this paper is posed on a batch production plant of known structure, denominated i_0, which establishes time windows $d \in D$ in which each customer $i \in IC$ can request delivery of their orders. In this way, a mutual agreement is generated between the plant and the customers, in which the company commits to deliver the orders that the customers have placed for each time window d during the interval $[a_d, c_d]$. The orders, which may contain a mix of products $p \in P$, are processed in a single-stage batch facility that has multiple non-identical $u \in U$ units operating in parallel, which have different capacities, processing times and costs depending on the type of product being processed. These parameters are problem data and are represented under the following nomenclature: $capmin_{up}$ and $capmax_{up}$ define the maximum and minimum capacities of unit u for processing product p, fpt_{up} is the fixed processing time for each batch of product p in unit u, and fpc_{up} is the processing cost of a batch of product p in unit u.

Each customer can place an order, consisting of one or more products, in each time window d. Thus, the amount of product p demanded by customer i in time window d, represented by dem_{ipd}, is a parameter of the problem. To satisfy customer demand and achieve better equipment utilization, each batch $b \in B_p$ of product p can be used to satisfy different orders demanding that product. Since the number of batches of each product, as well as their sizes, are variables of the problem, appropriate quotas must be proposed a priori for them in order to ensure the optimality of the solution and to facilitate the solution of the problem. Considering the total demand of product p over all time windows, and the minimum capacity required for processing product p in unit u, it is possible to calculate the maximum number of batches of product p that can be processed in unit u to satisfy customer orders, NB_{up}^{UP}, by the expression (1).

$$NB_{up}^{UP} = \left\lceil \frac{\sum_{i \in IC} \sum_{d \in D} dem_{ipd}}{capmin_{up}} \right\rceil \qquad \forall\, u \in U, \forall\, p \in P \tag{1}$$

The parameter $nb_p = max_{u \in U}\{NB_{up}^{UP}\}$ $\forall\, p \in P$ is used to define the set $B_p = \{b_1, b_2, \ldots, b_{nb_p}\}$, which denotes the set of batches assigned to the product p. For production scheduling decisions, a continuous time representation based on time slots is used. This approach is based on dividing the time horizon into predefined time slots that act as intervals in which batches are assigned to be processed in units. Identifying the number of slots to be proposed for each unit in the plant is not a trivial task, since the number of batches of all products is a variable in the problem. So, a clear challenge of this time representation is to propose an appropriate number of slots for each unit. The start and duration of the slots are unknown and are also part of the problem solution. Obviously, taking into account the possibility of the extreme case in which all product batches are processed in the same unit, this number is defined as the summation in p of the parameters NB_{up}^{UP}. However, depending on the processing times of products and the proposed time windows for product delivery to customers, this parameter may result in an overestimation, which directly affects the performance of the problem. Consequently, considering the information regarding the problem data, a tighter value for this parameter can be proposed, which is detailed in subsection 2.1.

To deliver orders, the plant has a heterogeneous fleet of vehicles $v \in V$, which are grouped into different types of vehicles $vt \in VT$, according to shared characteristics such as capacities and transportation costs. For a vehicle to be used, the total amount of goods to be transported must not exceed the maximum allowable capacity, cap_{vt}, and in turn, must exceed a minimum occupancy percentage denoted by prc_{vt}. Each vehicle can make only one trip within the time horizon. Each vehicle starts and ends at the plant, may include visiting one or more customers, and must make only one stop at each customer location visited. The distance and travel time between different customers and the plant, $dist_{ii'}$ and $tv_{ii'}$ with i, $i' \in IC \cup \{i_0\}$/ $i \neq i'$, are problem data. The loading and unloading time of the products is independent of the order size and is included in the travel time. The transportation cost is divided into two components: a fixed cost, ftc_{vt}, which corresponds to the use of the *vt*-type vehicle, and a variable cost, vtc_{vt}, which depends on the distance traveled by the *vt*-type vehicle. In each vehicle, orders from different customers can be loaded, but each order associated with a time window must be delivered completely in a single vehicle. Thus, partial deliveries are not allowed. In this way, in each vehicle, the batches or parts of batches required to complete each order distributed by this truck are consolidated. For each selected vehicle, the sequence of customer-time windows to be visited is a model variable. These routing decisions are represented by the notion of immediate precedence.

2.1 Estimation of proposed slots

The proposed approach divides the time horizon into predefined time intervals to represent the allocation of batches to the processing units. These slots can be either synchronous or asynchronous: in the synchronous representation, the slots are identical in all units, which facilitates coordination but may limit flexibility in scheduling. In contrast, the asynchronous representation allows each unit to have specific slots adapted to its requirements. Thus, flexibility is increased to adjust operation times, although computational complexity is also greater. In this paper, production decisions are represented using the asynchronous time-slots approach. In addition, the following assumptions must be taken into account:

- Each slot of a specific unit can process only one batch at a time.
- The slot length will be zero if no batch is assigned to it.
- The number of slots to be used is unknown and may differ for each unit.

In the previous subsection, the number of time slots in the extreme case where all batches are processed in one unit was calculated (see Eq. (1)). However, depending on the data, this value may result in an overestimation that affects the performance of the problem. Therefore, an optimization model is proposed that allows to calculate the number of slots adjusted to the real conditions of the integrated problem. The integer variable SP_{up} represents the number of feasible slots that can be allocated in each unit u of the product p. Thus, Eq. (2) states that the time required to process the slots in each unit must be less than or equal to the largest upper bound of all time windows minus the shortest travel time between the plant and a customer, denoted by MTD. The parameter NB_{up}^{UP}, calculated previously, sets the upper bound for the variable SP_{up}, since the number of batches of product p to postulate in each unit u must not exceed this value, Eq. (3). Finally, the objective function, Eq. (4), maximizes the sum of the combination of slots.

$$\sum_{p \in P} fpt_{up} SP_{up} \leq max_{d \in D} \{c_d\} - MTD \qquad \forall\, u \in U \tag{2}$$

$$SP_{up} \leq NB_{up}^{UP} \qquad \forall\, u \in U, \forall\, p \in P \tag{3}$$

$$Max \sum_{u \in U} \sum_{p \in P} SP_{up} \tag{4}$$

Using the results of this model, the parameter $ns_u = max_{p \in P} \{SP_{up}\}$ $\forall\, u \in U$ is required to define the set $L_u = \{l_1, l_2, \ldots, l_{ns_u}\}$, which represents the proposed allocated slots for each unit.

2.2 Decisions involved

Under the above assumptions, the model determines the number, size, allocation, sequencing and detailed timing of the batches to be processed in each unit of the plant, the vehicles to be used, the allocation of orders to each transport unit, the route and the precise timing of the visits to customers by each vehicle, in order to minimize the total cost of production and distribution operations. To explain the complexity of the decisions that are considered simultaneously in the addressed problem, Fig. 1 is presented. In the illustrated example, three products p_1, p_2 and p_3), represented by the colors green, orange and purple, respectively, are considered. Each product has a total demand that must be processed in batches and, therefore, number and size of batches must be determined. In the first box on the left (batching problem), the division of demand into batches for each product is shown. The demands for products p_1 and p_3 are divided into three batches (b_1, b_2 and b_3), while the demand for product p_2 is divided into two batches (b_1 and b_2). In addition, each batch must be assigned to a unit and each unit must process one batch at a time. Therefore, as shown in the central part of Fig. 1 (production problem), the allocation and sequence of batches in each processing unit, as well as the time required for production, must be determined. In the schematic example, unit u_1 processes 2 batches of product p_1 and one batch of p_3. In the case of unit u_2, it processes one batch of product p_1, two batches of p_3 and all batches of p_2.

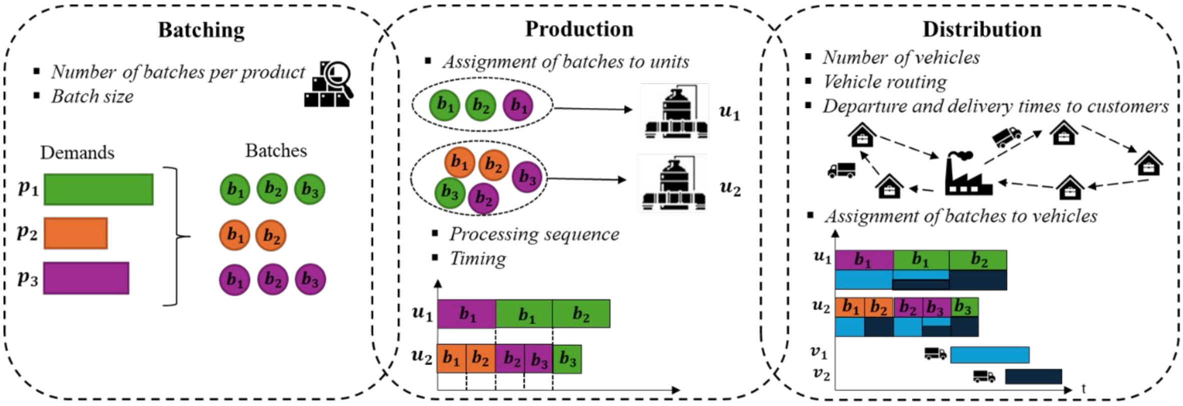

Fig. 1. Diagram of the decisions involved in the three problems studied

Finally, the decisions related to the distribution problem are shown in the box on the right. In the example, two vehicles (v_1 and v_2), represented in light blue and blue colors, respectively, are used. Using these colors, the figure shows under each batch how they are assigned to the vehicles. The batches can be assigned completely, as is the case with batch b_1 of product p_3 to vehicle v_1, or partially, as illustrated with batch b_1 of product p_1, which is assigned in part to vehicle v_1 and vehicle v_2. Although the sequence of customers to be visited by each vehicle is not exemplified in this figure, it is also part of the decisions to be made in this problem.

It is essential to highlight the importance of the integration of the three problems: the size of each batch is subject to the dimensions of the unit where it will be processed and, consequently, the number of batches of each product to be produced must be determined simultaneously. The allocation of batches to units, as well as the sequence of processing in each unit is vital for the synchronization between the order completion times and the departure times of the vehicles to which these batches will be partially or fully allocated to be shipped to the corresponding customers. The delivery sequence of each vehicle is determined by the assignment of customers to each vehicle. All these decisions are solved simultaneously to provide the optimal production and distribution plan that satisfies demand requirements while minimizing operating costs. Although solving batching, production and distribution problems simultaneously is computationally and operationally challenging, it has advantages and benefits that reward the invested effort.

3. Model formulation

The model presented in this section considers two groups of constraints: batching and production, and distribution scheduling. In the first group, the following restrictions are included: i) *number and size of batches*: equations that allow determining the number and size of batches to be produced to guarantee the satisfaction of the demand, complying with the minimum and maximum quotas for the size of each batch, ii) *assignment of batches to units:* equations that guarantee that each batch must be processed by a unit, iii) *batch sequencing:* restrictions that model the batch processing sequence in each unit, and iv) *times*: equations that determine the processing time of each batch in the unit in which it is assigned, which avoid overlapping in the processing of batches in each unit. For the second group: (v) *vehicle capacity*: equations that ensure that the amount of final product allocated to each selected vehicle does not exceed its maximum capacity and meets the minimum required, (vi) *assignment of batches to vehicles*: constraints that ensure that each vehicle transports exactly what is requested by each customer assigned to its route, vii) *vehicle routing*: equations that establish the precedence relationships between the customers visited by each vehicle, and viii) *departure and arrival times to customers*: constraints that ensure that vehicles do not leave the plant until all assigned batches are processed, as well as equations that ensure that orders are delivered within the time limits for which they were requested. In addition, taking into account the combinatorics of the problem, constraints are incorporated to reduce alternative solutions. Finally, the objective function minimizes the cost of production and distribution.

3.1 Batching and production scheduling constraints

The continuous variable BS_{bp} is defined, which represents the size of batch b of product p. Eq. (5) guarantees that exactly the total demanded by customers is produced.

$$\sum_{b \in B_p} BS_{bp} = \sum_{i \in IC} \sum_{d \in D} dem_{ipd} \qquad \forall\, p \in P \tag{5}$$

The binary variable X_{bpul} is equal to 1 if batch b of product p is assigned to slot l in unit u, otherwise zero. Through the Eqs. (6) and (7), the maximum and minimum dimensions for the size of the batches are established.

$$BS_{bp} \le \sum_{u \in U} \sum_{l \in L_u} capmax_{up} X_{bpul} \qquad \forall\, p \in P, \forall\, b \in B_p \tag{6}$$

$$BS_{bp} \geq \sum_{u\in U}\sum_{l\in L_u} capmin_{up} X_{bpul} \qquad \forall\, p \in P, \forall\, b \in B_p \tag{7}$$

Eq. (8) ensures that batches must be assigned to at most one slot of a unit, while Eq. (9) determines that no more than one batch of product can be processed in each slot of a unit at a time.

$$\sum_{u\in U}\sum_{l\in L_u} X_{bpul} \leq 1 \qquad \forall\, p \in P, \forall\, b \in B_p \tag{8}$$

$$\sum_{p\in P}\sum_{b\in B_p} X_{bpul} \leq 1 \qquad \forall\, u \in U, \forall\, l \in L_u \tag{9}$$

To avoid alternative solutions, additional constraints on slots and batches allocation are added in this formulation. Thus, Eqs. (8) and (9) ensure that the slots of each unit and the batches of the same product type are used in increasing order, respectively.

$$\sum_{p\in P}\sum_{b\in B_p} X_{bpul} \geq \sum_{p\in P}\sum_{b\in B_p} X_{bpul+1} \qquad \forall\, u \in U, \forall\, l, l+1 \in L_u \tag{10}$$

$$\sum_{u\in U}\sum_{l\in L_u} X_{bpul} \leq \sum_{u\in U}\sum_{l\in L_u} X_{b+1pul} \qquad \forall\, p \in P, \forall\, b, b+1 \in B_p \tag{11}$$

On the other hand, Eq. (12) eliminates symmetrical solutions by requiring that the denomination of batches of the same product follow an ascending order with respect to the units used.

$$\sum_{l' \in L_u \,/\, l'<l} X_{bpul'} + \sum_{u'<u}\sum_{l''\in L_{u'}} X_{bpu'l''} \geq X_{b+1pul} \quad \forall\, p \in P, \forall\, b, b+1 \in B_p, \forall\, u \in U, \forall\, l \in L_u \tag{12}$$

A continuous time representation is used to determine the exact times at which events occur. In this context, the continuous variables ST_{ul} and FT_{ul} are defined, representing the start and end time of slot l processing in unit u, respectively. Eq. (13) computes the completion time of slot l in unit u, which is obtained by adding the fixed processing time of product p in unit u to the start time of processing in slot l in that unit.

$$FT_{ul} = ST_{ul} + fpt_{up} X_{bpul} \qquad \forall\, p \in P, \forall\, b \in B_p, \forall\, u \in U, \forall\, l \in L_u \tag{13}$$

The overlap between the processing times of different slots in each unit is avoided by using Eq. (14). Moreover, if no batch is assigned to slot l+1 of unit u, its initial time is equal to the final time of slot l (Eq. (15)). M_1 is a sufficiently large scalar.

$$FT_{ul} \leq ST_{ul+1} \qquad \forall\, u \in U, \forall\, l, l+1 \in L_u \tag{14}$$

$$FT_{ul} \geq ST_{ul+1} - M_1 \sum_{p\in P}\sum_{b\in B_p} X_{bpul+1} \qquad \forall\, u \in U, \forall\, l, l+1 \in L_u \tag{15}$$

3.2 Distribution scheduling constraints

Given that each batch can be partially used to satisfy orders that are distributed on different vehicles, the binary variable R_{bpv} indicating whether batch b of product p is assigned to vehicle v, and the continuous variable QT_{bpv} representing the number of units of that batch that are loaded on the vehicle, are defined. In addition, the binary variable W_v indicates whether the vehicle v is used or not. The set VVT_{vt} containing the vehicles of type vt is defined. The Eqs. (16) and (17) ensure that the total number of units assigned to each vehicle meets the minimum required capacity of the vehicle, but does not exceed its maximum capacity, respectively. In both equations, the left-hand term is multiplied by the scalar α_p, which is a factor that determines the relationship between units of final product and their weight, since the capacity of each vehicle is given in units of weight.

$$\sum_{p\in P}\sum_{b\in B_p} \alpha_p QT_{bpv} \geq prc_{vt} cap_{vt} W_v \qquad \forall\, vt \in VT, \forall\, v \in VVT_{vt} \tag{16}$$

$$\sum_{p\in P}\sum_{b\in B_p} \alpha_p QT_{bpv} \leq cap_{vt} W_v \qquad \forall\, vt \in VT, \forall\, v \in VVT_{vt} \tag{17}$$

Eq. (18) ensures that all units of batch b of product p are assigned to vehicles. On the other hand, Eq. (19) ensures that the variable QT_{bpv} takes the value zero if batch b of product p is not assigned to vehicle v ($R_{bpv} = 0$). However, if the binary variable R_{bpv} equals one, as stated in Eq. (20), the vehicle must load at least one unit of the product. To reduce the search space, the number of batches per product that can be assigned to each vehicle is bounded by Eq. (21).

$$\sum_{v \in V} QT_{bpv} = BS_{bp} \qquad \forall\, p \in P, \forall\, b \in B_p \tag{18}$$

$$QT_{bpv} \leq max_{u \in U} \{capmax_{up}\} R_{bpv} \qquad \forall\, p \in P, \forall\, b \in B_p, \forall\, v \in V \tag{19}$$

$$QT_{bpv} \geq R_{bpv} \qquad \forall\, p \in P, \forall\, b \in B_p, \forall\, v \in V \tag{20}$$

$$\sum_{b \in B_p} R_{bpv} \leq \left\lceil \frac{\sum_{b \in B_p} QT_{bpv}}{min_{u \in U}\{ capmin_{up}\}} \right\rceil \qquad \forall\, p \in P, \forall\, v \in V \tag{21}$$

The binary variable Z_{idv} determines whether customer order i requested for time window d is assigned to vehicle v. Since partial deliveries of orders are not allowed, the order requested by each customer in a time window must be shipped in a single vehicle (Eq. (22)). In addition, it is not allowed to visit the same customer more than once during a vehicle trip. This implies that the customer can only be visited once with the same vehicle for the delivery of an order in a specific time window, or, in case two time windows coincide temporally, both orders can be delivered simultaneously (Eq. (23)).

$$\sum_{v \in V} Z_{idv} = 1 \qquad \forall\, i \in IC, \forall\, d \in D \tag{22}$$

$$Z_{idv} + Z_{id'v} \leq 1 \qquad \forall\, i \in IC, \forall\, d, d' \in D/(d+1<d') \vee (d'=d+1 \wedge c_d \neq a_{d+1}), \forall\, v \in V \tag{23}$$

Eq. (24) guarantees that all customers who are visited by the vehicle v are delivered the total amount of units that compose their orders.

$$\sum_{p \in P} \sum_{b \in B_p} QT_{bpv} = \sum_{i \in IC} \sum_{d \in D} dem_{ipd} Z_{idv} \qquad \forall\, v \in V \tag{24}$$

For routing decisions which establish the order of customers to be visited in the same route, the following binary variables are defined: ZP_{iv} equals 1 if customer i is the first to be visited in the route of vehicle v, $Y_{ii'v}$ equals 1 if customer i is delivered immediately before customer i' with vehicle v, and finally ZU_{iv} whose value is 1 if customer i is the last in the route of vehicle v, and zero otherwise. Each used vehicle must be assigned one customer who is the first to visit and one who is the last (Eqs. (25) and (26)).

$$\sum_{i \in IC} ZP_{iv} = W_v \qquad \forall\, v \in V \tag{25}$$

$$\sum_{i \in IC} ZU_{iv} = W_v \qquad \forall\, v \in V \tag{26}$$

However, if customer i is the first or the last customer to be visited on the route of vehicle v, the requested goods must be delivered in at least one time window (Eqs. (27) and (28)). Similarly, Eqs. (29) and (30) propose the same assumption for precedence-succession relationships.

$$ZP_{iv} \leq \sum_{d \in D} Z_{idv} \qquad \forall\, i \in IC, \forall\, v \in V \tag{27}$$

$$ZU_{iv} \leq \sum_{d \in D} Z_{idv} \qquad \forall\, i \in IC, \forall\, v \in V \tag{28}$$

$$Y_{ii'v} \leq \sum_{d \in D} Z_{idv} \qquad \forall\, i, i' \in IC/\, i \neq i', \forall\, v \in V \tag{29}$$

$$Y_{i'iv} \leq \sum_{d \in D} Z_{idv} \qquad \forall\ i, i' \in IC/\ i \neq i', \forall\ v \in V \tag{30}$$

Customer i may be visited by vehicle v first, or just after another customer i'. Similarly, customer i can be assigned to the last position in the route of vehicle v, or just before another customer i'. The formulations of these constraints are given by Eqs. (31)-(34).

$$ZP_{iv} + \sum_{i' \in IC/i \neq i'} Y_{i'iv} \geq Z_{idv} \qquad \forall\ i \in IC, \forall\ d \in D, \forall\ v \in V \tag{31}$$

$$ZP_{iv} + \sum_{i' \in IC/i \neq i} Y_{i'iv} + Z_{idv} \leq 2 \qquad \forall\ i \in IC, \forall\ d \in D, \forall\ v \in V \tag{32}$$

$$ZU_{iv} + \sum_{i' \in IC/i \neq i'} Y_{ii'v} \geq Z_{idv} \qquad \forall\ i \in IC, \forall\ d \in D, \forall\ v \in V \tag{33}$$

$$ZU_{iv} + \sum_{i' \in IC/i \neq i'} Y_{ii'v} + Z_{idv} \leq 2 \qquad \forall\ i \in IC, \forall\ d \in D, \forall\ v \in V \tag{34}$$

Eq. (35) guarantees that the precedence relationship between two customers of the same vehicle must be unique.

$$Y_{ii'v} + Y_{i'iv} \leq 1 \qquad \forall\ i, i' \in IC/\ i \neq i', \forall\ v \in V \tag{35}$$

For the time constraints, the following continuous variables are defined: DT_v is used to represent the departure time of vehicle v, and DET_{iv} is used to indicate the arrival time at customer i in vehicle v. The departure time of each vehicle must be greater than or equal to the latest end time of the batches assigned to it. To model this condition, a Big-M constraint is used, where the scalar M_2 represents the maximum upper bound among all the time windows considered (Eq. (36)). Note that this constraint is redundant if any of the variables X_{bpul} or R_{bpv} is null. On the other hand, Eq. (37) orders vehicles of the same type according to the departure time, while Eq. (38) determines that if vehicle v must deliver at least one order for time window d, the departure time of this vehicle must be less than the upper bound of this window.

$$DT_v \geq FT_{ul} - M_2\left(2 - X_{bpul} - R_{bpv}\right) \qquad \forall\ p \in P, \forall\ b \in B_p, \forall\ u \in U, \forall\ l \in L_u, \forall\ v \in V \tag{36}$$

$$DT_v - DT_{v+1} \leq M_2\ (2 - W_v - W_{v+1}) \qquad \forall\ vt \in VT, \forall\ v, v+1 \in VVT_{vt} \tag{37}$$

$$DT_v \leq c_d\ W_v + M_2(1 - Z_{idv}) \qquad \forall\ i \in IC, \forall\ d \in D, \forall\ v \in V \tag{38}$$

The departure time of each vehicle determines the delivery time to each customer. If customer i is the first customer to visit on vehicle route v, the delivery time is defined by Eqs. (39) and (40), where i_0 represents the production plant. These constraints guarantee that the delivery time to the first customer of each vehicle must be equal to the sum of the travel time from the plant to the customer plus the travel start time of vehicle v. In the case where $ZP_{iv} = 0$, appropriate scalars are used to ensure that these equations are redundant ($M_3 = M_2 + max_{i \in IC}\{tv_{ii_0}\}$).

$$DET_{iv} \geq DT_v + tv_{i_0 i} - M_3(1 - ZP_{iv}) \qquad \forall\ i \in IC, \forall\ v \in V \tag{39}$$

$$DET_{iv} \leq DT_v + tv_{i_0 i} + M_2(1 - ZP_{iv}) \qquad \forall\ i \in IC, \forall\ v \in V \tag{40}$$

Similar to Eqs. (39) and (40), Eqs. (41) and (42) determine the delivery time to customers who are immediate predecessors (not the first on the route). In this case, the delivery time to customer i' is calculated as the delivery time of its predecessor plus the travel time between them. Note that through Eqs. (39)-(42), it is guaranteed that there are no idle times in the travel time of each vehicle.

$$DET_{i'v} \geq DET_{iv} + tv_{ii'} - M_3(1 - Y_{ii'v}) \qquad \forall\ i, i' \in IC/\ i \neq i', \forall\ v \in V \tag{41}$$

$$DET_{i'v} \leq DET_{iv} + tv_{ii'} + M_2(1 - Y_{ii'v}) \qquad \forall\ i, i' \in IC/\ i \neq i', \forall\ v \in V \tag{42}$$

Customer orders must be delivered within the limits of the corresponding time window d (Eqs. (43) and (44)).

$$DET_{iv} \geq a_d\, W_v - M_2(1 - Z_{idv}) \qquad \forall\, i \in IC, \forall\, d \in D, \forall\, v \in V \qquad \textbf{(43)}$$

$$DET_{iv} \leq c_d\, W_v + M_2(1 - Z_{idv}) \qquad \forall\, i \in IC, \forall\, d \in D, \forall\, v \in V \qquad \textbf{(44)}$$

If no orders are delivered to customer *i* in vehicle *v*, the delivery time to customer *i* in vehicle *v* must be zero (Eq. (45)). Orders with precedence relation can only be delivered if the travel time between customers does not generate a delivery delay (Eq. (46)).

$$DET_{iv} \leq M_2 \sum_{d \in D} Z_{idv} \qquad \forall\, i \in IC, \forall\, v \in V \qquad \textbf{(45)}$$

$$tv_{ii'}Y_{ii'v}\text{-}max_{i\in IC}\{tv_{ii'}\}(2\text{-}Z_{idv}\text{-}Z_{i'd'v}) \leq c_{d'} - a_d \qquad \forall\, i, i'\in IC/\, i{\neq}i', \forall\, d, d' \in D/\, d{<}d', \forall\, v \in V \qquad \textbf{(46)}$$

The variables R_{bpv}, Z_{idv}, $Y_{ii'v}$, DT_v and DET_{iv} take the value zero if the vehicle *v* is not used (Eqs. (47)-(51)). To reduce the search space, Eq. (52) ensures vehicles of the same type are used in ascending order. M_4 is the difference between M_2 and the minimum travel time between a customer and the plant.

$$R_{bpv} \leq W_v \qquad \forall\, p \in P, \forall\, b \in B_p, \forall\, v \in V \qquad \textbf{(47)}$$

$$Z_{idv} \leq W_v \qquad \forall\, i \in IC, \forall\, d \in D, \forall\, v \in V \qquad \textbf{(48)}$$

$$Y_{ii'v} \leq W_v \qquad \forall\, i, i' \in IC\, /\, i{\neq}i', \forall\, v \in V \qquad \textbf{(49)}$$

$$DT_v \leq M_4\, W_v \qquad \forall\, v \in V \qquad \textbf{(50)}$$

$$DET_{iv} \leq M_2 W_v \qquad \forall\, i \in IC, \forall\, v \in V \qquad \textbf{(51)}$$

$$W_v \geq W_{v+1} \qquad \forall\, vt \in VT, \forall\, v, v{+}1 \in VVT_{vt} \qquad \textbf{(52)}$$

3.3 Objective function

The objective function is the minimization of the total operating cost given by production and distribution costs (Eq. 53).

$$\begin{aligned}\text{Min}\ & \sum_{p\in P}\sum_{b\in B_p}\sum_{u\in U}\sum_{l\in L_u} fpc_{up}X_{bpul}\ + \\ & \sum_{vt\in VT}\sum_{v\in VVT_{vt}} ftc_{vt}W_v + \sum_{i\in IC}\sum_{vt\in VT}\sum_{v\in VVT_{vt}} vtc_{vt}dist_{i_0 i}ZP_{iv} + \sum_{i\in IC}\sum_{i'\in IC}\sum_{vt\in VT}\sum_{v\in VVT_{vt}} vtc_{vt}dist_{ii'}Y_{ii'v} \\ & + \sum_{i\in IC}\sum_{vt\in VT}\sum_{v\in VVT_{vt}} vtc_{vt}dist_{ii_0}ZU_{iv}\end{aligned} \qquad \textbf{(53)}$$

For production, a fixed processing cost is considered for the production of each batch, depending on the product and the unit in which it is produced. For distribution, a fixed cost for using the vehicle and a variable cost depending on the distance traveled are considered.

4. Illustrative examples

Three examples are presented in this section to show the capabilities of the proposed approach. The first one is of small dimension, but useful to assess the simultaneous solution of batching and production and distribution scheduling problems. The second and third examples are medium size problems, where the number of customers, time windows, units and vehicles has been increased compared to the first example, which strongly impacts the computational performance of the model. In all examples, the coefficient α_p, which represents the conversion factor from product units to kg, is assumed to be 4.75 for p_1, 4.0 for p_2 and 4.25 for p_3. Table 1 shows the number of customers, time windows, orders, products, units and vehicles considered for each of the three examples, followed by the number of equations, variables and CPU time to reach the optimal solution (0% GAP). The models were coded and implemented by gurobipy, a Python-based implementation of Gurobi v.10.0.0 (Gurobi

Optimization, LLC and Python Software Foundation, 2022; Gurobi Optimization, LLC, 2023) on a PC with an Intel Core i7 processor, 3.60 GHz and 32 GB of RAM.

Table 1
Parameters and computational statistics of examples

	Example 1	Example 2	Example 3
Number of customers	4	7	8
Time windows	2	3	4
Number of orders	8	12	16
Number of products	3	3	3
Number of proposed units	2	3	3
Number of proposed vehicles	7	7	9
Constraints	6030	8751	20721
Number of binary variables	791	1083	2311
Number of continuous variables	388	657	1158
CPU time (sec)	21	32	280

4.1 Example 1

The first example considers a batch production plant operating with two non-identical units in parallel, where a total of eight orders required by four different customers must be processed. Each order is composed of a mixture of three different products and is associated with one of two time windows d_1 and d_2, whose intervals (in hours) are: [9, 11] and [11, 13], respectively. For transportation, three types of vehicles are available, where the first type has three units and the other two types have two vehicles. The example data are presented in Tables 2-5: the demand for each time window is presented in Table 2, the processing times in hours, the minimum and maximum capacity in units of final product, and the processing costs in $ per batch, of each unit for each product are shown in Table 3. Next, Table 4 shows the distance, in kilometers, between the plant and the customers, and finally, the minimum and maximum capacity for each type of vehicle, as well as the cost of the vehicles, are presented in Table 5. Although this is a small example, this is a demanding scheduling, since all customers require all products in all time windows.

Table 2
Customer demands for Example 1

Customer	Demand: dem_{ipd} (u)	
	d_1: [9,11] (h)	d_2: [11,13] (h)
i_1	p_1: 100 p_2: 100 p_3: 100	p_1: 150 p_2: 100 p_3: 50
i_2	p_1: 50 p_2: 100 p_3: 160	p_1: 150 p_2: 150 p_3: 50
i_3	p_1: 60 p_2: 100 p_3: 50	p_1: 50 p_2: 100 p_3: 90
i_4	p_1: 100 p_2: 50 p_3: 160	p_1: 50 p_2: 100 p_3: 150

Table 3
Unit parameters for Example 1

Unit	Processing time fpt_{up} (h/batch)			Unit capacity						Processing cost fpc_{up} ($/batch)		
				$capmin_{up}$ (u)			$capmax_{up}$ (u)					
	p_1	p_2	p_3	p_1	p_2	p_3	p_1	p_2	p_3	p_1	p_2	p_3
u_1	1	1.5	1	90	180	135	100	200	150	350	460	390
u_2	1	2	1.5	135	135	60	150	150	100	410	450	400

Table 4
Distance between nodes for Example 1

Nodes	i_0	i_1	i_2	i_3	i_4
i_0	0	140	160	170	72
i_1	140	0	400	70	230
i_2	160	400	0	120	150
i_3	170	70	120	0	175
i_4	72	230	150	175	0

Table 5
Vehicle capacities and costs for Example 1

Type of vehicle	Minimum capacity (kg)	Maximum capacity cap_{vt} (kg)	Fixed distribution cost ftc_{vt} ($)	Variable distribution cost vtc_{vt} ($/km)
vt_1 (v_1, v_2 v_3)	1225	1500	15	2
vt_2 (v_4, v_5)	2400	3000	18.75	2.5
vt_3 (v_6, v_7)	2800	4000	21	2.8

Tables 6 and 7 and Fig. 2 show the optimal solution found. Table 6 contains production details, such as the time slots used in each unit, their start and end times, the batches of product processed in each slot, the size of the batches, the percentages of unit utilization per batch processed and the assignment to the vehicles. Similarly, Table 7 shows the results of the distribution, such as the number of vehicles used according to type, the departure time of each vehicle, the route to be traveled (customers to be visited), the time of delivery to each customer visited, and the percentage of vehicle utilization.

Table 6
Batching and production scheduling details for Example 1

Unit	Slot	ST_{ul} (h)	FT_{ul} (h)	Batch	Product	Size (u)	Usage (%)	Assignment to vehicles (u)		
								v_4	v_6	v_7
u_1	l_1	0	1.5	b_1	p_2	200	100		200	
	l_2	1.5	3	b_2	p_2	200	100		100	100
	l_3	3	4.5	b_3	p_2	200	100			200
	l_4	4.5	6	b_4	p_2	200	100	150		50
	l_5	6	7	b_1	p_3	135	90		135	
	l_6	7	8	b_2	p_3	135	90			135
	l_7	8	9	b_3	p_3	135	90			135
	l_8	9	10	b_4	p_3	135	90	135		
u_2	l_1	0	1	b_1	p_1	144	96		144	
	l_2	1	2.5	b_2	p_1	144	96		144	
	l_3	2.5	4	b_3	p_1	144	96		22	122
	l_4	4	5.5	b_4	p_1	143	95	15		128
	l_5	5.5	6.5	b_5	p_3	90	90		65	25
	l_6	6.5	7.5	b_6	p_3	90	90	85		5
	l_7	7.5	8.5	b_5	p_1	135	90	135		
	l_8	8.5	9.5	b_7	p_3	90	90	90		
Production cost: $6650										

Table 7
Distribution scheduling details for Example 1

Vehicle	Type	Departure time (h)	Customer	Time window	Delivery time (h)	Usage (%)
v_4	vt_2	10.1	i_4	$d_1 - d_2$	11	88
v_6	vt_3	8	i_3	d_1	10.1	88
			i_1	$d_1 - d_2$	11	
v_7	vt_3	9	i_2	$d_1 - d_2$	11	97
			i_3	d_2	12.5	
Distribution cost: $7181						

To illustrate the optimal solution found, the Gantt chart is presented in Fig. 2.

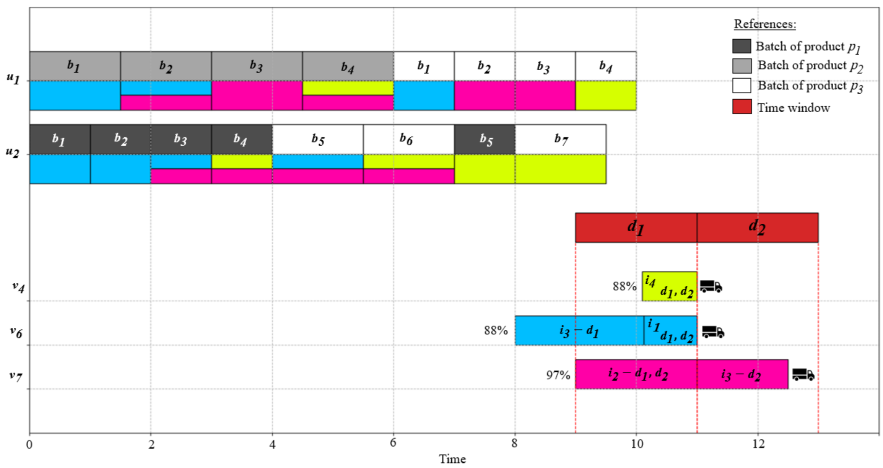

Fig. 2. Production and distribution schedule for Example 1

In the center of the figure, the time windows d_1 and d_2, marked in red, separate the graph into two parts. The upper part shows the production schedule for each processing unit (u_1 and u_2), and the lower part shows the distribution schedule for each vehicle used (v_4, v_6 and v_7). For a better understanding of the information illustrated in each part, the unit u_1 and the vehicle v_6 are explained in detail. The first row of u_1 shows that 8 slots are processed, 4 of which are batches of product p_2 (grey product), while the remaining 4 are of product p_3 (white product). Note that the length of the slots refers to the time it takes to process them, it does not refer to the batch size. Then, in the row immediately below, it is possible to see in which vehicle each batch is loaded. For example, batch b_1 of product p_2 is assigned to vehicle v_6 (light blue vehicle), while batch b_2 of product p_2 is assigned in parts to vehicle v_6 and vehicle v_7 (pink vehicle). For the distribution solution, vehicle v_6 leaves at 8 a.m., with 88% of its capacity occupied, first visits customer i_3, to which it delivers the order placed for time window d_1 at 10.1 a.m. Then, the vehicle v_6 travels to customer i_1, to which it delivers the order for time windows d_1 and d_2 at 11 a.m.

4.2 Example 2

In this example, three non-identical units are considered that operate in parallel to process three different products. Seven customers may request orders for three time windows d_1, d_2 and d_3, whose intervals (in hours) are: [7, 9], [9, 11] and [11, 13], respectively. For the distribution of twelve orders, three types of vehicles are available, where two types have two units each, and one has three vehicles. As in Example 1, the data for Example 2 are presented in Tables 8-11, while the optimal solution is detailed in Tables 12 and 13 and Fig. 3.

Table 8
Customer demands for Example 2

Customer	Demand: dem_{ipd} (u)		
	d_1: [7,9] (h)	d_2: [9,11] (h)	d_3: [11,13] (h)
i_1	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0	p_1: 100 p_2: 0 p_3: 100
i_2	p_1: 50 p_2: 100 p_3: 60	p_1: 50 p_2: 155 p_3: 50	p_1: 0 p_2: 0 p_3: 0
i_3	p_1: 0 p_2: 0 p_3: 0	p_1: 25 p_2: 37 p_3: 30	p_1: 25 p_2: 74 p_3: 0
i_4	p_1: 100 p_2: 50 p_3: 0	p_1: 0 p_2: 100 p_3: 50	p_1: 74 p_2: 20 p_3: 60
i_5	p_1: 0 p_2: 0 p_3: 0	p_1: 30 p_2: 30 p_3: 30	p_1: 300 p_2: 150 p_3: 200
i_6	p_1: 50 p_2: 50 p_3: 0	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0
i_7	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0	p_1: 100 p_2: 100 p_3: 200

Table 9
Unit parameters for Example 2

Unit	Processing time fpt_{up} (h/batch)			Unit capacity						Processing cost fpc_{up} ($/batch)		
				$capmin_{up}$ (u)			$capmax_{up}$ (u)					
	p_1	p_2	p_3	p_1	p_2	p_3	p_1	p_2	p_3	p_1	p_2	p_3
u_1	2	3	1	120	110	115	130	125	130	350	460	390
u_2	2	3	2	140	120	115	150	140	130	410	450	400
u_3	3	1	2.5	130	115	120	140	130	135	450	390	380

Table 10
Distance between nodes for Example 2

Nodes	i_0	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_0	0	200	120	170	72	80	400	80
i_1	200	0	400	70	230	150	400	400
i_2	160	400	0	120	200	96	96	350
i_3	170	70	120	0	175	300	200	200
i_4	72	230	200	175	0	104	88	450
i_5	80	150	96	300	104	0	160	340
i_6	400	400	96	200	88	160	0	390
i_7	80	400	350	200	450	340	390	0

Table 11
Vehicle capacities and costs for Example 2

Type of vehicle	Minimum capacity (kg)	Maximum capacity cap_{vt} (kg)	Fixed distribution cost ftc_{vt} ($)	Variable distribution cost vtc_{vt} ($/km)
vt_1 (v_1, v_2)	1400	2000	22.4	2.1
vt_2 (v_3, v_4)	2400	3000	20	1.9
vt_3 (v_5, v_6, v_7)	2625	3500	16	1.5

Table 12
Batching and production scheduling details for Example 2

Unit	Slot	ST_{ul} (h)	FT_{ul} (h)	Batch	Product	Size (u)	Usage (%)	Assignment to vehicles (u)			
								v_1	v_5	v_6	v_7
u_1	l_1	0	2	b_1	p_1	122	94		122		
	l_2	2	4	b_2	p_1	122	94		122		
	l_3	4	6	b_3	p_1	120	92	56	64		
	l_4	6	7	b_1	p_3	130	100			130	
	l_5	7	9	b_4	p_1	120	92	100			20
	l_6	9	10	b_2	p_3	130	100				130
	l_7	10	11	b_3	p_3	125	96	125			
u_2	l_1	0	2	b_5	p_1	140	93			140	
	l_2	2	4	b_6	p_1	140	93				140
	l_3	4	6	b_4	p_3	130	100		130		
	l_4	6	8	b_5	p_3	130	100	10			120
	l_5	8	10	b_7	p_1	140	93				140
u_3	l_1	0	1	b_1	p_2	130	100		130		
	l_2	1	2	b_2	p_2	130	100		81	49	
	l_3	2	3	b_3	p_2	130	100			130	
	l_4	3	5.5	b_6	p_3	135	100	65		70	
	l_5	5.5	6.5	b_4	p_2	126	97			126	
	l_6	6.5	7.5	b_5	p_2	120	92				120
	l_7	7.5	8.5	b_6	p_2	115	88				115
	l_8	8.5	9.5	b_7	p_2	115	88	100			15
Production cost: $7710											

Table 13
Distribution scheduling details for Example 2

Vehicle	Type	Departure time (h)	Customer	Time window	Delivery time (h)	Usage (%)
v_1	vt_1	11	i_7	d_3	12	86
v_5	vt_3	6.5	i_4	d_1	7.4	81
			i_6	d_1	8.5	
			i_3	d_2 - d_3	11	
			i_1	d_3	11.9	
v_6	vt_3	7.5	i_2	d_1 - d_2	9	87
			i_5	d_2	10.2	
			i_4	d_3	11.5	
v_7	vt_3	10.1	i_4	d_2	11	99
			i_5	d_3	12.3	
Distribution cost: $8041						

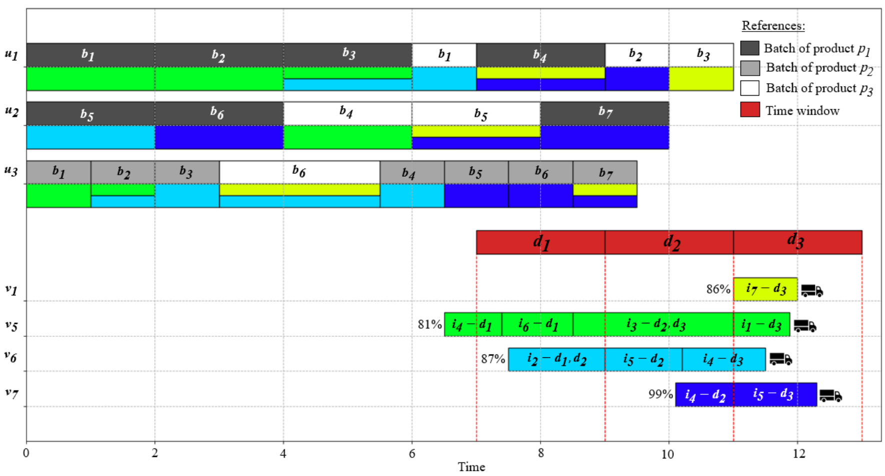

Fig. 3. Production and distribution schedule for Example 2

4.3 Example 3

As in Example 2, Example 3 considers a medium-scale case, where three non-identical parallel units that process three different products are considered. On the demand side, eight customers may request orders in four different time windows d_1, d_2, d_3 and d_4, of duration [7,8], [8,9], [10,11] and [11,12], respectively. For order distribution, three types of vehicles are available, which have two, four and three vehicles associated with them, respectively. The model input data for the instance under study are presented in Tables 16-19 and the optimal solution is shown in Tables 20, 21 and 22 and Fig. 4.

Table 14
Customer demands for Example 3

Customer	Demand: dem_{ipd} (u)			
	d_1: [7, 8] (h)	d_2: [8, 9] (h)	d_3: [10, 11] (h)	d_4: [11, 12] (h)
i_1	p_1: 50 p_2: 50 p_3: 50	p_1: 50 p_2: 50 p_3: 50	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0
i_2	p_1: 50 p_2: 50 p_3: 50	p_1: 50 p_2: 50 p_3: 50	p_1: 0 p_2: 0 p_3: 0	p_1: 90 p_2: 90 p_3: 90
i_3	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0	p_1: 60 p_2: 50 p_3: 40	p_1:50 p_2: 50 p_3: 50
i_4	p_1: 60 p_2: 60 p_3: 60	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0	p_1: 50 p_2: 50 p_3: 50
i_5	p_1: 20 p_2: 30 p_3: 0	p_1: 0 p_2: 0 p_3: 70	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0
i_6	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0	p_1: 100 p_2: 100 p_3: 100
i_7	p_1: 30 p_2: 30 p_3: 30	p_1: 90 p_2: 90 p_3: 90	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0
i_8	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 0 p_3: 0	p_1: 0 p_2: 50 p_3: 0	p_1: 200 p_2: 220 p_3: 230

Table 15
Unit parameters for Example 3

Unit	Processing time fpt_{up} (h/batch)			Capacity units						Processing cost fpc_{up} (\$/batch)		
				$capmin_{up}$ (u)			$capmax_{up}$ (u)					
	p_1	p_2	p_3	p_1	p_2	p_3	p_1	p_2	p_3	p_1	p_2	p_3
u_1	1.1	1	2	100	80	100	110	90	110	370	420	400
u_2	1	1.5	1	70	90	105	80	100	115	400	420	395
u_3	1.5	1	1	80	80	90	95	110	100	450	430	400

Table 16
Distance between nodes for Example 3

Nodes	i_0	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8
i_0	0	80	96	120	160	250	168	100	76
i_1	80	0	250	94	80	150	160	260	230
i_2	96	250	0	150	200	40	92	250	175
i_3	120	94	150	0	80	250	200	200	260
i_4	160	80	200	80	0	150	100	180	260
i_5	250	150	40	250	150	0	270	60	350
i_6	168	160	92	200	100	270	0	92	240
i_7	100	260	250	200	180	60	92	0	260
i_8	76	230	175	260	260	350	240	260	0

Table 17
Vehicle capacities and costs for Example 3

Type of vehicle	Minimum capacity (kg)	Maximum capacity cap_{vt} (kg)	Fixed distribution cost ftc_{vt} (\$)	Variable distribution cost vtc_{vt} (\$/km)
vt_1 (v_1, v_2)	375	500	15.3	1.26
vt_2 (v_3, v_4, v_5, v_6)	1050	1500	21.25	2.1
vt_3 (v_7, v_8, v_9)	2600	3250	17	1.75

Table 18
Batching and production scheduling details for Example 3

Unit	Slot	ST_{ul} (h)	FT_{ul} (h)	Batch	Product	Size (u)	Usage (%)
u_1	l_1	0	1.1	b_1	p_1	110	100
	l_2	1.1	2.1	b_1	p_2	90	100
	l_3	2.1	3.2	b_2	p_1	110	100
	l_4	3.2	4.3	b_3	p_1	110	100
	l_5	4.3	5.4	b_4	p_1	110	100
	l_6	5.4	6.5	b_5	p_1	110	100
	l_7	6.65	7.75	b_6	p_1	109	99
	l_8	7.75	8.95	b_7	p_1	109	99
	l_9	8.95	10.05	b_8	p_1	102	93
u_2	l_1	0	1	b_1	p_3	115	100
	l_2	1	2	b_2	p_3	115	100
	l_3	2	3	b_9	p_1	80	100
	l_4	3	4	b_3	p_3	115	100
	l_5	4	5	b_4	p_3	115	100
	l_6	5	6	b_5	p_3	115	100
	l_7	6	7	b_6	p_3	115	100
	l_8	7	8	b_7	p_3	115	100
	l_9	8	9	b_8	p_3	105	91
u_3	l_1	0	1	b_2	p_2	110	100
	l_2	1	2	b_3	p_2	110	100
	l_3	2	3	b_4	p_2	110	100
	l_4	3	4	b_5	p_2	110	100
	l_5	4	5	b_9	p_3	100	100
	l_6	5	6	b_6	p_2	110	100
	l_7	6	7	b_7	p_2	110	100
	l_8	7.05	8.05	b_8	p_2	98	89
	l_9	8.05	9.05	b_9	p_2	86	78
	l_{10}	9.05	10.05	b_{10}	p_2	86	78
Production cost: \$11210							

Table 19
Distribution scheduling details for Example 3

Vehicle	Type	Departure time (h)	Customer	Time window	Delivery time (h)	Usage (%)
v_3	vt_2	6.5	i_7	d_1	7.7	89
			i_5	d_2	8.5	
			i_2	d_2	9	
v_4	vt_2	7.7	i_7	d_2	9	78
v_5	vt_2	9.9	i_6	d_4	12	87
v_7	vt_3	5	i_4	d_1	7	100
			i_1	d_1 - d_2	8	
			i_2	d_4	11.1	
v_8	vt_3	6.2	i_2	d_1	7.4	87
			i_5	d_1	7.9	
			i_3	d_3 - d_4	11	
			i_4	d_4	12	
v_9	vt_3	10.1	i_8	d_3 - d_4	11	93
Distribution cost: \$18628						

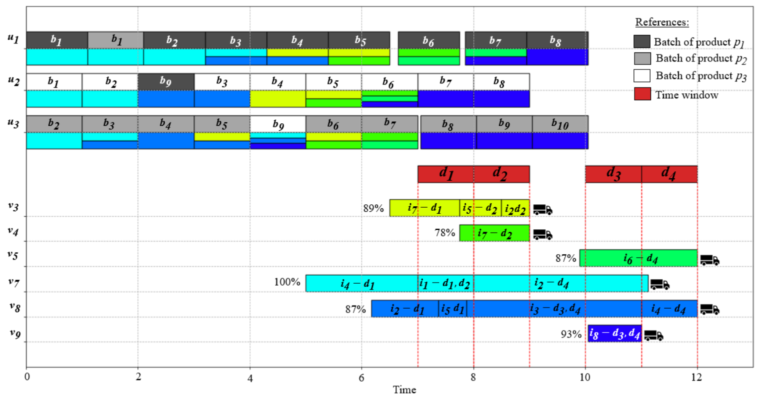

Fig. 4. Production and distribution schedule for Example 3

Table 20
Assignment of batches to vehicles in Example 3

Batch	Product	Size (u)	Assignment to vehicles (u)					
			v_3	v_4	v_5	v_7	v_8	v_9
b_1	p_1	110				110		
b_1	p_2	90				90		
b_2	p_1	110				110		
b_3	p_1	110				30	80	
b_4	p_1	110	40				70	
b_5	p_1	110	40	70				
b_6	p_1	109		20	89			
b_7	p_1	109			11			98
b_8	p_1	102						102
b_1	p_3	115				115		
b_2	p_3	115				115		
b_9	p_1	80						80
b_3	p_3	115					115	
b_4	p_3	115	115					
b_5	p_3	115	35	80				
b_6	p_3	115		10	100			5
b_7	p_3	115						115
b_8	p_3	105						105
b_2	p_2	110				110		
b_3	p_2	110				50	60	
b_4	p_2	110					110	
b_5	p_2	110	50				60	
b_9	p_3	100				20	75	5
b_6	p_2	110	30	80				
b_7	p_2	110		10	100			
b_8	p_2	98						98
b_9	p_2	86						86
b_{10}	p_2	86						86

5. Integrated vs. Sequential approach

Is the integrated approach more cost-effective than the sequential approach? Is it worth the computational effort? To answer these questions, the three case studies presented in the previous section are evaluated using a hierarchical methodology. The subsection 5.1 describes the sequential approach used, highlighting the constraints associated with each stage. The next subsection shows the solution obtained by solving each of the examples using the sequential approach. Subsequently, a detailed comparison of the approaches used in the three examples is presented in the subsection 5.3.

5.1 Two-stage hierarchical approach

The problem under study is solved in two consecutive stages. In the first one, decisions related to the number and size of batches to be processed, the allocation of these batches to units, the processing sequence and the production schedule are determined. All these decisions are taken to minimize the total cost of production. Thus, in the first stage, the problem subject to Eqs. (5)-(15) is solved and the production cost is minimized (Eq. (54)). It is important to note that no constraints related to distribution logistics are considered in the first stage of the hierarchical approach.

$$\text{Min} \sum_{p\in P}\sum_{b\in B_p}\sum_{u\in U}\sum_{l\in L_u} fpc_{up}X_{bpul} \tag{54}$$

Stage 2 consists of determining all decisions related to distribution logistics, such as the number of vehicles to be used, the customers visited by each vehicle, and the time of departure and delivery to the customers. On this basis, there are different alternatives to solve this problem. The first and most basic is to assume that all decisions related to the production problem have already been taken, so the problem to be solved in Stage 2 is minimize the distribution cost (Eq. (55)) subject to constraints (16)-(52). This is the most direct procedure, but it is much more likely to obtain solutions with very poor performance, or even infeasible.

$$\begin{aligned}\text{Min} &\sum_{vt\in VT}\sum_{v\in VVT_{vt}} ftc_{vt}W_v + \sum_{i\in IC}\sum_{vt\in VT}\sum_{v\in VVT_{vt}} vtc_{vt}dist_{i_0 i}ZP_{iv} + \\ &\sum_{i\in IC}\sum_{i'\in IC}\sum_{vt\in VT}\sum_{v\in VVT_{vt}} vtc_{vt}dist_{ii'}Y_{ii'v} + \sum_{i\in IC}\sum_{vt\in VT}\sum_{v\in VVT_{vt}} vtc_{vt}dist_{ii_0}ZU_{iv}\end{aligned} \tag{55}$$

Another alternative is to use the optimal solution to the batching and production scheduling problems, but limiting their effects. The optimal value obtained for production scheduling depends on the assignment of the determined batches to the units. In this new alternative, it is assumed that these decisions remain, but the times at which batch processing begins in the units where they were assigned can be varied. In this way, the previous solution is used, but with greater flexibility that allows for better performance in distribution scheduling. Thus, in this paper, Stage 2 solves the problem that minimizes the distribution cost (Eq. (55)) subject to Eqs. (5)-(52). Note that the variables for assigning batches to units are set according to the values obtained in Stage 1. To illustrate the process of the proposed approach, a flowchart is presented in Fig. 5.

The solution obtained by addressing the problem using the sequential approach can be: (i) feasible, where its value could be equal to or worse than that obtained with the integrated approach; or (ii) infeasible, which indicates that, according to the production schedule found, it is not possible to fulfill the delivery of orders to customers within the established time windows. This last alternative is now usual, taking into account that the production optimization has been carried out without taking into account the commitments with customers and the available vehicles. Therefore, this leads to companies holding stock levels to meet demand or, in the worst case, backorders are generated.

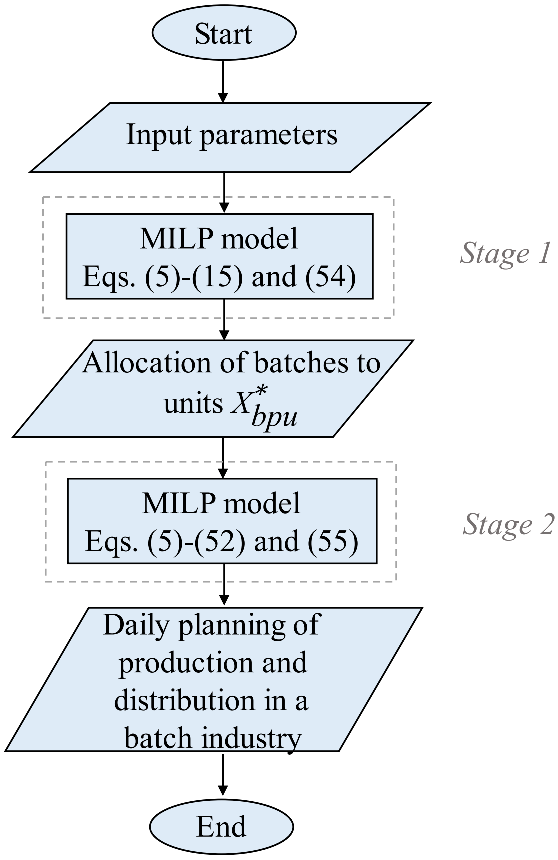

Fig. 5. Flowchart of the two-stage hierarchical approach

5.2 Sequential resolution of illustrative examples

In the following, each of the examples discussed in this paper is solved using the approach described in the previous subsection. The main objective is to compare and analyze whether the effort of solving the two activities simultaneously is worthwhile. Table 21 summarizes the key information for an effective comparison of both approaches. For each example, the objective function value and resolution time for each approach are shown, as well as the percentage improvement obtained with the integrated approach. The table shows that, although the computational times are slightly longer in the integrated approach, they are reasonable for the operational and complex problem being solved, while the percentage improvement indicates that the effort is really valuable and justified. A brief description of each example is provided below along with its respective Gantt chart.

Table 21
Comparison of examples under sequential and integrated approaches

	Sequential approach		Integrated approach		% improvement
	Objective function ($)	CPU time (sec)	Objective function ($)	CPU time (sec)	
Example 1	16594	5	13831	21	17
Example 2	17586	15	15751	32	10
Example 3	33570	48	29838	280	16

5.2.1 Example 1

Fig. 6 presents in detail the solution of Example 1 using the two-stage hierarchical approach. The total cost is \$16594, where \$6230 corresponds to production and \$10364 to distribution. The computation time was 5 seconds. Gantt diagram shows that batches are processed in the most economically convenient units. This production scheduling implies that the vehicles must follow direct routes to a single customer in order to deliver on time (with the exception of v_6). Thus, it is evident that not consolidating small orders in a single vehicle generates significantly higher transportation costs compared to the opposite approach.

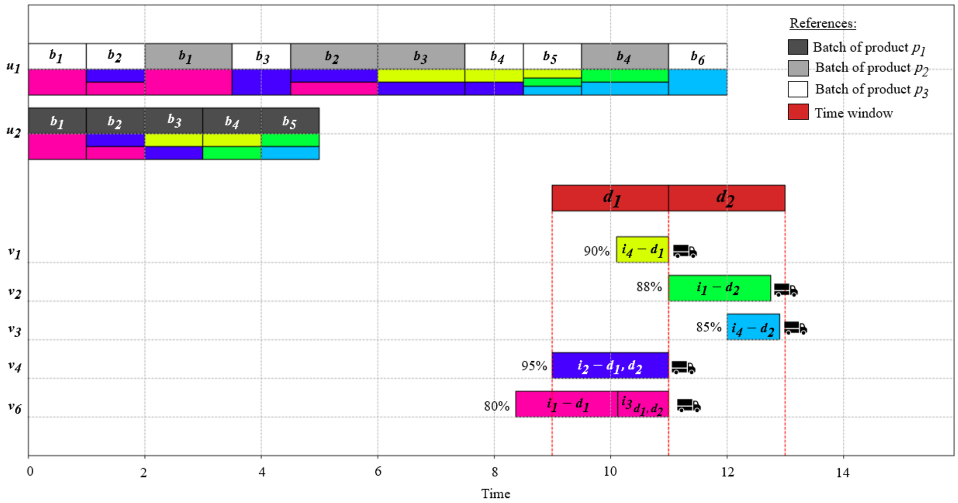

Fig. 6. Gantt chart of Example 1 solved using the sequential approach

5.2.2 Example 2

The solution obtained using the sequential approach in Example 2 is shown in Fig. 7. In terms of production, the tendency is to produce all batches in the most economical units: batches of product p_1 in unit u_1, batches of product p_3 in unit u_2 and batches of product p_2 in unit u_3. However, due to processing times and time window limits, not all product batches can be processed in the most convenient units. Thus, batch b_7 of product p_1 is processed in u_2 and batch b_6 of p_3 in unit u_3. The total cost of the solution is \$17586, consisting of \$7620 production cost and \$9966 distribution cost. This case study was solved in 15 seconds.

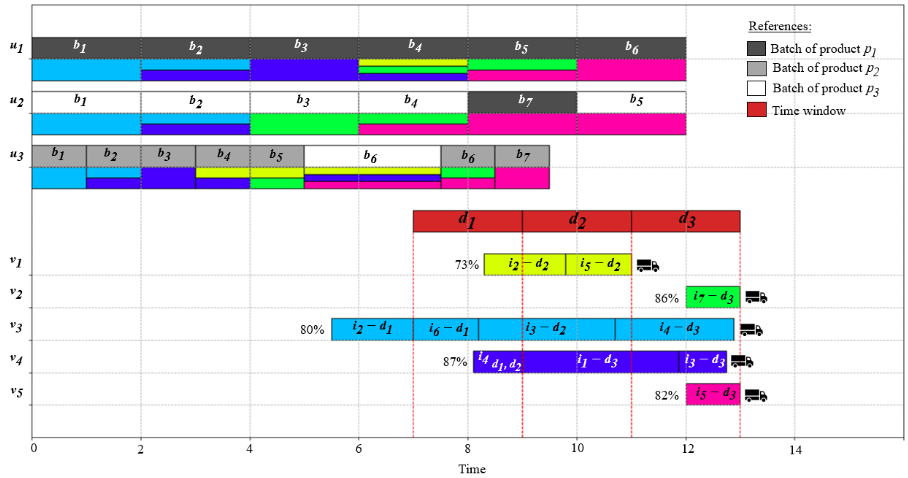

Fig. 7. Gantt chart of Example 2 solved using the sequential approach

5.2.3 Example 3

Finally, Fig. 8 shows the solution of Example 3 using the sequential approach that was obtained in 48 seconds. In this case, the production cost is \$11160 and the distribution cost is \$22410, yielding an operating cost of \$33570.

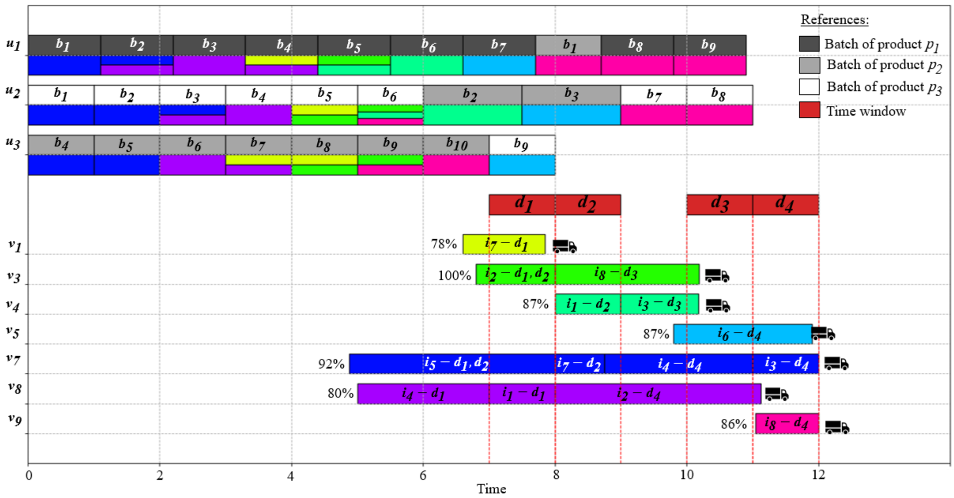

Fig. 8. Gantt chart of Example 3 solved using the sequential approach

5.3 Analysis of results

Analyzing the solutions obtained in the examples, it is possible to identify different patterns and behaviors depending on the applied solution approach. In all cases, the sequential approach presents an initial advantage in terms of production cost due to a better allocation of batches to equipment, where units are used prioritizing their low cost. However, more efficient production planning often leads to less effective distribution logistics. In this case, with no finished product inventory, vehicle routes must be organized according to batch completion times. This significantly reduces flexibility and generally results in direct shipments to customers. As a consequence, the total distance traveled at the end of the day tends to be greater in order to meet demands, and, more importantly, a greater number of vehicles is usually required. These situations not only generate additional costs, but also lead to an inefficient utilization of vehicle capacity, the emission of more kilograms of greenhouse gases, a larger fleet and a higher number of drivers, etc. The integrated approach stands out for its ability to improve efficiency in the supply chain operations, deliver orders on time, optimize vehicle capacity and increase customer satisfaction. Figs. 9, 10 and 11 present a comparison of production and distribution costs in the three examples studied. In each figure, for both the integrated and sequential approaches, production costs are shown in pink and distribution costs in light blue. As can be seen, the common denominator is that, in the sequential approach, the production cost tends to be slightly lower compared to the integrated approach, while distribution costs are significantly higher. The numerical solutions highlight the considerable important savings that can be achieved by solving the integrated approach. The improvements obtained in the case studies analyzed are 17%, 10% and 16% for Examples 1, 2 and 3, respectively.

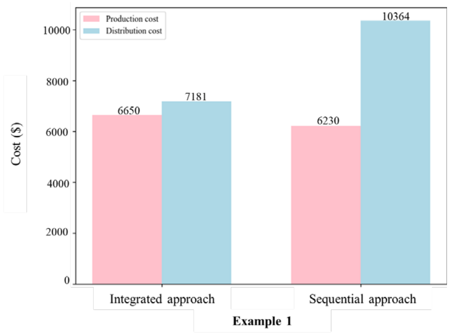

Fig. 9. Cost comparison for Example 1

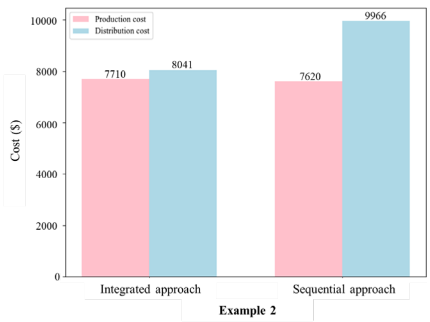

Fig. 10. Cost comparison for Example 2

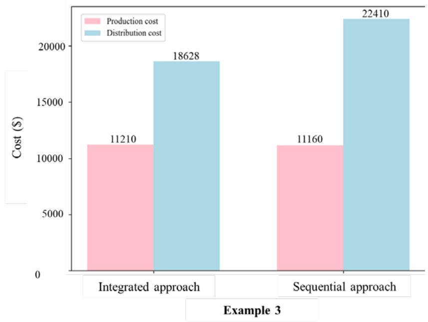

Fig. 11. Cost comparison for Example 3

6. Conclusions

In recent years, due to highly competitive environments, the need to respond to customer requirements has become an essential goal for companies. This context has promoted the production of customized products, which implies a major challenge for the production and distribution areas, as these products tend to have practically zero inventories. MTO production systems are not the only ones posing difficulties for industries, products with limited shelf life also present significant challenges for the production and distribution areas.

The integration of batching decisions in the context of IPDSP is an area that is largely unexplored in the literature. The introduction of the possibility of splitting orders into multiple batches has been shown to improve the efficiency of solutions, especially when perishable products are considered. However, few works have addressed this integration, and the proposed solutions do not always include an effective resolution of time constraints such as delivery dates or contemplate zero inventory policies, which are more complex but real situations.

According to this context, the main contribution of this work is the development of a MILP model that allows the simultaneous integration of batching, production and distribution scheduling decisions in a single-stage multi-product batch plant. In the proposed approach, each customer can place several orders whose deliveries can be associated to different time windows proposed by the company and mutually agreed with the customers. The developed model simultaneously determines where, how, and when the batches are processed and how they are loaded and shipped in the selected vehicles, in order to reduce production and distribution costs and satisfy customer delivery in a timely manner. The proposed approach finds the optimal solution in reasonable computational times.

In order to show the capabilities of the developed model and the importance of optimally solving the problem in contrast to a sequential approach, three examples were analyzed. The exact approach proves to be clearly superior to the sequential approach. The improvements of more than 10% observed in the examples indicate that the integrated approach offers optimal solutions, overcoming the limitations of the sequential approach and providing a significant advantage in terms of computational efficiency and performance.

Acknowledgments

The authors appreciate financial provided by Argentine research institutions, namely the National Council for Scientific and Technical Research (CONICET, by its initials in Spanish), the National Agency for Scientific and Technical Promotion (ANPCyT, by its initials in Spanish) and the National Technological University (UTN, by its initials in Spanish). Support was granted in the framework of projects PIP 2021-2023 GI- 11220200101401CO, PICT-2020-01644 and PID SITCAFE0008410TC, respectively.

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