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The non-stationary stochastic lot-sizing with joint replenishment under (R, S) policy and the heuristics

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This study investigates for the first time the non-stationary stochastic lot-sizing problem involving multi-dealer joint replenishment under the policy (R, S) without fill rate constraints. The planning horizon for each dealer is divided into the replenishment cycle series, accounting for the lead time associated with each joint replenishment cycle. A shortest path model is developed. Through mathematical analysis, the safety stock variables are eliminated, and the multiple variables are reduced to replenishment variables only. The stochastic problem is converted to the deterministic dynamic lot-sizing through expectation analysis. Furthermore, the MLS-MRS heuristic is proposed based on Robinson's Left-Right shift (LS-RS) heuristic by adding a module, the positive cost-saving family shifts. This algorithm improves the optimal solution and notably greatly increases the search speed.

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1. Introduction

Since the pioneering research of Wagner and Whitin (1958), lot-sizing with dynamic demands has become an important issue in inventory management practice and theory. The lot-sizing problem (LSP) has gradually evolved from a single-item problem to a multi-item problem, resulting in dynamic-deterministic demands joint replenishment problem (DDJRP) initially proposed by Goyal (1973), i.e., demands deterministic but varying with time. Various formulations are developed for DJRP, an NP-complete problem (Joneja, 1990). First, the traditional formulation calculates the setup, the order size, and the holding costs period by period. Given the formulation, it successively raises the joint ordering policies mitigating multi-item impact approach (ter Haseborg, 1982), a combined branch-and-bound (BB) and dynamic programming (DP) procedure (Erenguc, 1988), an exserted BB technique complying with a greedy-add heuristic and a tight lower bound (Federgruen & Tzur, 1994), and a counterpart-the greedy drop heuristic (Boctor et al., 2004). Second, Fogarty and Barringer (1987) proposed the dynamic program formulation, assuming every item is produced each time. Subsequently, Silver E (1988) rescheduled some items into an earlier schedule to save on costs for improving the former. Robinson et al. (2007) proposed the heuristic assembling FB (Fogarty & Barringer, 1987) and SK heuristic (Silver E, 1988). Third, the shortest path formulation, proposed by Joneja (1990), involves the joint setup cost and generated cost of *N* independent single items. The latter is modeled by involving the individual setup, order, and holding costs in the order cycle, while the holding cost is formulated as the ending stock at the period multiplied by the unit cost . Fourth, the formulation by Robinson and Gao (1996) refers to both setup costs and order costs plus holding costs, whereas the unit inventory cost is designed to cumulate in the replenishment cycle and is solved by a dual-ascent based BB procedure. Moreover, the fifth model by Boctor et al. (2004) resembles the second, except that the holding cost is formulated by multiplying the unit period demand by the time interval (ordering timing to consumption timing). There are also other heuristics of solving DJRP. Iyogun (1991) developed two part–period balancing approaches. Robinson et al. (2007) proposed two forward-pass heuristics based on the different decision criteria proposed by Eisenhut (1975) and Lambrecht and Vanderveken (1979), respectively, as well as the Left-Right shift heuristic. Boctor et al. (2004) extended Silver–Meal heuristic. In addition, it exists the integration of the traditional heuristic into some metaheuristic, such as

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incorporating perturbation operations for iteration (Boctor et al., 2004) and simulated annealing algorithm with the Left-Right shift heuristic by Robinson et al. (2007). Since the stochastic demand was noticed by Silver (1978), the deterministic problem is gradually deepened to the stochastic demands, involving the demands varying with time but stochastic uncertainty. Stochastic uncertainty more features the environmental reality. To solve the uncertainty of the stochastic LSP, Bookbinder and Tan (1988) defined three policies:"the static-uncertainty (SU)", "the static-dynamic uncertainty (SDU)", and "the dynamic-uncertainty (DU)". The SU policy fixes the order schedule and size prior to the planning horizon; however, the order size balances inventory holding cost and order cost weakly. In contrast, the DU policy fixes the replenishment point and the maximum stock level (*s,S*) in schedule planning. The order cycles are not fixed, and orders occur randomly during the horizon, which can potentially lead to the system's nervousness. Accordingly, the SDU policy covers the shortage of the SU and DU policies while preserving their advantages. In the SDU policy, the order cycle *R* and the maximum stock level *S* are fixed before the execution. During the execution of the planning period, the order is carried out according to the previously fixed replenishment horizon. If the stock on hand does not meet the maximum stock level, the order is placed to *S*.

Bookbinder and Tan (1988) proposed a two-phase heuristics to the SDU policy problem under service-level constraints. Firstly, the replenishment cycles are determined using a static-uncertainty strategy. Secondly, an equivalent linear model is proposed under the fixed replenishment periods, calculating the optimal replenishment quantity of each replenishment cycle. Tarim and Kingsman (2004) proposed a MIP model under service-level constraints for a single-item problem, which determines the strategy (*R,S*) in a single step. Order cycles cannot be treated independently, thereby complementing Bookbinder and Tan 's special case. Tarim and Kingsman (2006) developed a equivalent MIP model without any constraints. Tarim et al. (2011) then provided an better relaxation for the MIP approach and proved that the relaxation approach can mostly obtain optimal solutions. Otherwise, the costs incurred by the solution are used as strict lower bounds if the solution is not feasible and the solution is modified to be feasible, resulting in an upper bound. Özen et al. (2012) investigated the SD policy under the known replenishment cycles and developed two algorithms: approximation and approximation heuristic for the proposed dynamic programming model. Tunc et al. (2018) developed an new model that calculates each cost within each replenishment cycle as a unit segment and then accumulated the cost of each replenishment cycle segment. In addition, a novel dynamic cut generation approach was proposed. Especially where the demand is non-stationary stochastic, the study of this problem is still studied by some researchers, Xiang et al. (2018), Visentin et al. (2021),Ma et al. (2022), and Visentin et al. (2023). They studied the (*R,Q*), (*R,S*), and (*S,s*) for LSP under the non-stationary stochastic demands based on the definition in Bookbinder and Tan (1988). While the DJRP literature referenced above examines the replenishment of multiple products, demands are deterministic in advance. Although the aforementioned uncertainty problem pertains to non-stationary and stochastic demand, the literatures focus on a single item and the known service level constrain. If the members of the supply system are affiliated to one entity, the decision can be optimized without considering service level constraints. Moreover, most of the lot-sizing literature assumes, for ease of research, that there is no lead time and that the cost of holding inventory per period is a linear function of ending stock. However, in the case of the larger unit period, it is interesting to see whether the lead time or the ending inventory method of calculating the inventory holding cost affects the decisions.

There are few studies of multi-item lot sizing with joint replenishment under non-stationary stochastic demands. Due to the SDU policy over the DU policy in causing weak system tension, the lot-sizing problem with the three-factor combination of multiple items, joint replenishment, and non-stationary stochastic demands under the SDU policy is studied in this paper. It is more complicated and NP-hard. We make the following contributions.

(1) The multi-dealer non-stationary stochastic problem with lead time under SDU policy (*R,S*) is studied for the first time and modeled as the shortest path formulation. This problem is essentially stochastic, but is a further investigation of the dynamic deterministic LSP of a single item and considers the lead time as well as the weaker system tension strategy (*R,S*). Therefore, the problem is more realistic. The problem is converted to the 0-1 problem by mathematically analyzing and eliminating the safety stock factor variable and to a deterministic problem by expectation analysis.
(2) We propose an improved Robison's Left-Right-shift heuristic algorithm that not only compares the cost savings of the individual shift to the family shift but also designs the positive-family shift—all shifts with positive individual-shift cost-savings, which greatly accelerates the search. The improved algorithm solves this paper's stochastic problem. It is also based on the dynamic deterministic LSP and is still applicable to this problem. It is an even better approach to the heuristic algorithm for this problem.

2. Model formulation

2.1 Assumptions and notation

The supply chain structure comprises one supplier and multiple dealers. The supplier is the distributor and must provide sufficient products for all dealers. The supplier adheres to the zero-stock policy, which entails the absence of any pre existing inventory. Upon the receipt of an order, the manufacturer will either produce the goods in question or the distributor will purchase them from a third party. It inevitably entails a certain lead time. The duration of lead times is known in advance. Each dealer experiences multiple periods. Demand per period is normally distributed, and the distributional characteristics vary with each period. Thereby, dealers face penalties for being out of stock due to stochastic demands. Dealers adopt the interval-review and order-up-to-position policy, (*R, S*). The problem is the static-dynamic uncertain rule (Tempelmeier, 2007), and it is solved by determining the replenishment timing and order size before the horizon. At the replenishment period, dealers conduct their inventory review. When the stock is below S, top up. It is noted that due to the varying attributes of the demand distribution across each period and multiple dealers, the review cycle *R* and the up-to-order level are not fixed, but rather dynamic. Consequently, each parameter is a vector variable, i.e. $R=(R_1, R_2, ...)$, $S=(S_1, S_2, ...)$, where numeric subscripts indicate the serial number of the orders.

Dealers who replenish on the identical period engage in joint replenishment (JR), and the supplier produces together. The supplier transports the products to each dealer by a one-to-one delivery policy. The lead time for each joint replenishment is known. The transportation time is neglected. Each dealer has the initial stock but meets the average demands of the period 1' lead time. The notations are shown below.

Indices
i: the dealer, i =1, 2, …, N.
t/l: the period, t=1, 2, …, T.
Parameters:
L: the unit period length (days).
π_t: the leading time (days).
D_{it}: the day demand in the period t (unit/day).
u_{it}: the average day demand rate of dealer i in period t (unit/day).
σ_{it}: the standard deviation of dealer i's day demand fluctuations in period t.
S: the major ordering costs per one order (dollar/order).
s_i: the minor ordering costs of dealer i per one order (dollar/order).
h_i: the unit inventory holding costs of dealer i (dollar/ (unit·day)).
I_i^0: the initial inventory of dealer i (unit).
$I_{i,t+\pi_{it}}$: the net stock of dealer i at time $t+\pi_{it}$ before the replenishment (unit).
c_i^p: the unit shortage costs of dealer i (dollar/unit).
Decision variables
x_{it}: the replenishment variables. If dealer i orders at period t, then 1. Otherwise, 0.
y_{itl}: the order covering periods variable. If an order covers demands over t,…, l, then 1. Otherwise, 0.
$z_{i,(t,l)}$: the desired safety stock factor of dealer i for the interval form period t to l.
R_{it}: the order-up-to-position of dealer i for replenishment cycle t (unit).
Q_{it}: the order size of dealer i at for replenishment cycle t.

2.2 Formulation

The overall costs include the setup, the inventory holding, and the shortage costs of all dealers in the horizon. Subsequently, the shortest path formulation for the total cost is proposed according to Kao (1979).

$$E(TC)=E\left(\sum_{t=1}^{T}S_t\delta_t+\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{l=t}^{T}\left(s_{it}x_{it}+y_{itl}(H_{itl}+P_{itl})\right)\right) \tag{1}$$

In the formula, H_{itl} and P_{itl} denote the stock holding and out-of-stock cost of the order cycle, respectively.

3.2.1. The inventory holding costs of the replenishment cycle

This paper employs a more precise approach by taking half of the curve inventory of the order cycle as the average stock, which is appropriate. The demand distribution of each dealer varies with the period, so the cycle and the stock level vary, as shown in Fig. 1. H_{itl} equals the cycle interval multiplied by the expected average stock in the interval, and the expected average stock is expressed as $E\left(\left(I_{i,t+\pi_{it}}+\max\left(I_{i,l+\pi_{il}},0\right)\right)h_i/2\right)$, $I_{i,t+\pi_{it}}$ and $I_{i,l+\pi_{il}}$ are the stock at $t+\pi_{it}$ and $l+\pi_{il}$ time, respectively. Hence the term $\mathrm{E}\left(\max\left(I_{i,l+\pi_{il}},0\right)\right)$ may be approximated $E\left(I_{i,l+\pi_{il}}\right)$. In consideration of the expected stockholdings and potential shortages, the subsequent analysis employs the safety stock factor z to hedge the inherent randomness of demand across different periods, and $z_{i,(t+\pi_{it},l+\pi_{il})}$ is defined as the desired safety stock factor for the time interval from the arrival timing $t+\pi_{it}$ to the next review time $t+\pi_{il}$, and simply denoted by z_{it}. Therefore, $E\left(\left(I_{i,t+\pi_{it}}+\max\left(I_{i,l+\pi_{il}},0\right)\right)/2\right)$ can be deserved by Eq. (2).

$$E\left(\left(I_{i,t+\pi_{it}}+Q_{it}+\max\left(I_{i,l+\pi_{il}},0\right)\right)/2\right)=\left(E\left(I_{i,t+\pi_{it}}\right)+E\left(I_{i,l+\pi_{il}}\right)\right)/2$$
$$=\left(E\left(I_{i,l+\pi_{il}}+\left(L-\pi_{it}\right)D_{it}+\sum_{j=t+1}^{l-1}LD_{itj}+\pi_{il}D_{il}\right)+E\left(I_{i,l+\pi_{il}}\right)\right)/2$$
$$=E\left(I_{i,l+\pi_{il}}\right)+E\left(\left(L-\pi_{it}\right)D_{it}+\sum_{j=t+1}^{l-1}LD_{itj}+\pi_{il}D_{il}\right)/2 \quad (1)$$
$$=z_{i,(t+\pi_{it},l+\pi_{il})}\delta_{i,(t+\pi_{it},l+\pi_{il})}+\left(\left(L-\pi_{it}\right)u_{it}+\sum_{j=t+1}^{l-1}Lu_{itj}+\pi_{il}u_{il}\right)/2$$
$$=z_{it}\delta_{i,(t+\pi_{it},l+\pi_{il})}+\left(\left(L-\pi_{it}\right)u_{it}+\sum_{j=t+1}^{l-1}Lu_{itj}+\pi_{il}u_{il}\right)/2$$

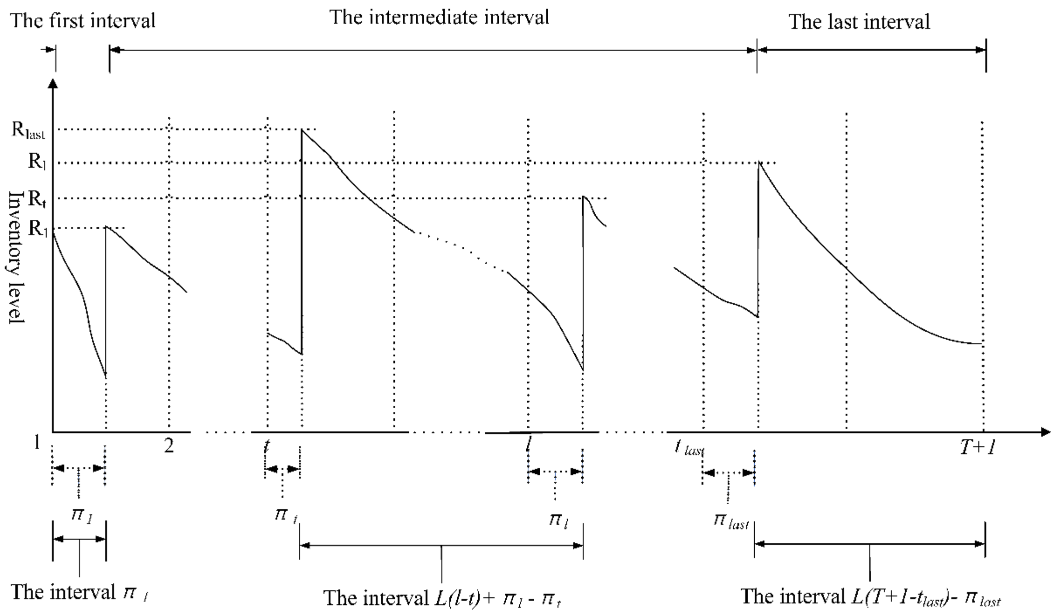

Fig. 1. Inventory changes of dealer i in the planning horizon

Given any decisions, the planning horizon of dealer i can be divided into a series of replenishment cycles, as Eq. (3), where y_{itl} equals 1. In this way, the inventory curve for each replenishment cycle is continuous, facilitating to calculate the average inventory for the interval.

$$\left(\pi_1,\cdots,L(l-t)+\pi_l-\pi_t,\cdots,L(T-t+1)-\pi_t\right) \quad (2)$$

Given the formula (3), the inventory holding costs of each dealer during the planning horizon are composed of the costs in period 1's lead time, the costs in the last interval, and the costs in the intermediate intervals. The intermediate replenishment intervals include multiple replenishment cycles, so the inventory holding costs are calculated from the following three sections. The length of each order cycle is denoted by *A(i,t)* except the interval π_1. The dealer i's average demands of *A(i,t)* is denoted by *B(i,t)*. Dealer i's safety stock of *A(i,t)* is denoted by *C(i,t)*.

(1) In lead time π_1 of dealer i's period 1, the initial inventory is I_i^0. The dealer i's average demands of the interval π_1 is formulated as $u_{i1}\pi_1$. Therefore, the expected initial inventory holding costs H_i^0 of the interval for dealer i is formulated as $h_i\left(I_i^0-u_{i1}\pi_1/2\right)\pi_1$.

(2) While t satisfies $y_{itT}=1$, t is the last order period. *A(i,t) equals* $L\left(T-t+1\right)-\pi_t$. The dealer i's average demands in the cycle, *B(i,t),* is formulated as $L\sum_{j=t}^{T}u_{ij}-u_{it}\pi_t$. Dealer i's standard deviation $\delta_{i,(t+\pi_i,l+\pi_l)}$ in the cycle, denoted by *C(i,t),* is formulated as $\sqrt{L\sum_{j=t}^{T}\sigma_{ij}^2-\pi_t\sigma_{it}^2}$. Therefore, the expected inventory holding costs in the cycle *A(i,t)* for dealer i are formulated

as $h_i\left(B(i,t)/2+z_{it}C(i,t)\right)A(i,t)$.

(3) While *t* is not the last replenishment period, and the order cycle *A(i,t) equals* $L(l-t)+\pi_l-\pi_t$. The average demands in each cycle, *B(i,t),* are formulated as $\sum_{j=t}^{l-1}Lu_{ij}+\left(\pi_l u_{il}-\pi_t u_{it}\right)$. The standard deviation *C(i,t)* is formulated as $\sqrt{L\sum_{j=t}^{l-1}\sigma_{ij}^2-\pi_t\sigma_{it}^2+\pi_l\sigma_{il}^2}$. Therefore, each replenishment cycle's expected inventory holding costs for dealer *i* in the intermediate cycle is formulated as $h_j\left(B(i,t)/2+z_{it}C(i,t)\right)A(i,t)$.

3.2.2. The shortage costs of the replenishment cycle

The shortage costs incurred by dealer *i* are also analyzed using the segmented intervals Eq. (3).

(1) In the lead time π_1, of dealer *i*'s period 1, the shortage costs P_i^0 are derived as Eq. (4).

$$p_i^0=c_i^pE\left(\max\left(D_{i1}\pi_{i1}-I_i^0,0\right)\right)=c_i^p\int_{I_i^0}^{\infty}\left(D-I_i^0\right)f\left(D\left(D_{i1},\pi_1\right)\right)dD=c_i^p\sigma_{i1}\sqrt{\pi_1}\int_{z_{i0}}^{\infty}\left(y-z_{i0}\right)f\left(y\right)dy \\ =c_i^p\sigma_{i1}\sqrt{\pi_1}\left(f\left(z_{i0}\right)-z_{i0}\left(1-F\left(z_{i0}\right)\right)\right) \tag{3}$$

where initial safety stock factor $z_{i0}=\left(I_i^0-u_{i1}\pi_1\right)\big/\left(\sigma_{i1}\sqrt{\pi_1}\right)$. The determination of z_{i0}, which is not a decision variable, relies on π_1 with given the known I_i^0. In Eq. (4), $f\left(\bullet\right)$ and $F\left(\bullet\right)$ denote the standard normal probability density function (PDF) and cumulative distribution function (CDF), respectively.

(2) In the last replenishment cycle $A(i,t)$ of dealer *i*, the shortage costs are derived as Eq. (5).

$$P_{itT}=c_i^pE\left(\max\left(\sum_{j=t-1}^{T}D_{ij}L+D_{it}\pi_t-I_{i,t+\pi_t}-Q_{it},0\right)\right)=c_i^p\int_{R_{it}}^{\infty}\left(D-R_{it}\right)f\left(D\left(D_{it},D_{i,t+1},\cdots,D_T,\pi_t\right)\right)dD \\ =c_i^pC(i,t)\int_{z_{it}}^{\infty}\left(y-z_{it}\right)f\left(y\right)dy=c_i^pC(i,t)\left(f\left(z_{it}\right)-z_{it}\left(1-F\left(z_{it}\right)\right)\right) \tag{4}$$

where *C(i,t) equals* $\sqrt{L\sum_{j=t}^{T}\sigma_{ij}^2-\pi_t\sigma_{it}^2}$, and R_{it} is the desired order-up-level at the timing $t+\pi_t$ and equals $I_{i,t+\pi_t}+Q_{it}$.

(3) In the middle cycle $A(i,t)$, the shortage costs of dealer *i* are derived as Eq. (6).

$$P_{itl}=c_i^pE\left(\max\left(\sum_{j=t-1}^{l}D_{ij}L+D_{it}\pi_t+D_{il}\pi_l-I_{t+\pi_t}-Q_{it},0\right)\right)=c_i^p\int_{R_{it}}^{\infty}\left(D-R_{it}\right)f\left(D\left(D_{it},\cdots,D_{il},\pi_l,\pi_t\right)\right)dD \\ =c_i^pC(i,t)\int_{Z_{it}}^{\infty}\left(y-z_{it}\right)f\left(y\right)dy=c_i^pC(i,t)\left(f\left(z_{it}\right)-z_{it}\left(1-F\left(z_{it}\right)\right)\right) \tag{5}$$

where *C(i,t) equals* $\sqrt{L\sum_{j=t}^{l-1}\sigma_{ij}^2-\pi_t\sigma_{it}^2+\pi_l\sigma_{il}^2}$.

3.2.3. The modified formulation

By modifying all costs formulation Eq.(1), the overall costs of all dealers during the planning horizon is formulated as Eq.(7).

$$E\left(TC\right)=\sum_{t=1}^{T}S_t\delta_t+\sum_{i=1}^{N}\left(H_i^0+P_i^0\right)+E\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{l=t}^{T}\left(s_{it}x_{it}+y_{itl}(H_{itl}+P_{itl})\right)\right) \tag{6}$$

subject to

a. $\delta_t\geq x_{it},t=1,2,\cdots T,i=1,2,\cdots N$

b. $\sum_{l=t}^{T}y_{itl}=1,t\in\left\{t\mid x_{it}=1\right\},i=1,2\cdots N$

c. $\exists t: \sum_{t=1}^{l} y_{itl} = 1, i = 1, 2, \cdots N$

d. $A(i,t) = \begin{cases} L(T-t+1) - \pi_t & \exists t: y_{itT} = 1 \\ L(l-t) + \pi_l - \pi_t & (t,l) \in \{(t,l) | y_{itl} = 1\} \setminus \{(t,T) | y_{itT} = 1\} \end{cases}$

e. $B(i,t) = \begin{cases} L\sum_{j=t}^{T} u_{ij} - u_{it}\pi_t & \exists t: y_{itT} = 1 \\ \sum_{j=t}^{l-1} Lu_{ij} + (\pi_l u_{il} - \pi_t u_{it}) & (t,l) \in \{(t,l) | y_{itl} = 1\} \setminus \{(t,T) | y_{itT} = 1\} \end{cases}$

f. $C(i,t) = \begin{cases} \sqrt{L\sum_{j=t}^{T} \sigma_{ij}^2 - \pi_t \sigma_{it}^2} & \exists t: y_{itT} = 1 \\ \sqrt{L\sum_{j=t}^{l-1} \sigma_{ij}^2 - \pi_t \sigma_{it}^2 + \pi_l \sigma_{il}^2} & (t,l) \in \{(t,l) | y_{itl} = 1\} \setminus \{(t,T) | y_{itT} = 1\} \end{cases}$

h. $H_{i0} + P_{io} = h_i \left(I_i^0 - u_{i1}\pi_1 / 2 \right) \pi_{i1} + c_i^p \sigma_{i1} \sqrt{\pi_1} \left(f(z_{i0}) - z_{i0} \left(1 - F(z_{i0}) \right) \right)$

i. $H_{itl} + P_{itl} = \begin{cases} h_i \left(B(i,t)/2 + z_{it} C(i,t) \right) A(i,t) + c_i^p C(i,t) \left(f(z_{it}) - z_{it} \left(1 - F(z_{it}) \right) \right) & t: y_{itl} = 1 \\ 0 \quad \text{otherwise} \end{cases}$

j. $z_{it^-} C(i,t^-) + E(Q_{it}) - B(i,t) = z_{it} C(i,t)$

L. $\delta_t \in \{0,1\} \quad t=1,2,\cdots T$

m. $x_{it} \in \{0,1\} \quad i = 1,2,\cdots N, t=1,2,\cdots T$

n. $y_{itl} \in \{0,1\} \quad i = 1,2,\cdots N, t=1,2,\cdots T, l = t,\cdots T$

o. $z_{it} > 0, \; i = 1,2,\cdots N, t=1,2,\cdots T,$

2.3 The optimal property analysis of the safe-stock factor z

Due to no service level constraint, the safe stock factor can be mathematically analyzed to minimize the objective function *TC*. The partial derivative of the variable z_{it} of the objective function is obtained $\partial TC / \partial z_{it} = h_i A(i,t) C(i,t) - C_i^p C(i,t) \left(1 - F(z_{it}) \right)$, and the solution procedure is shown in **Appendix A**. To $\partial TC / \partial z_{it} = 0$, $A(i,t) h_i = C_i^p \left(1 - F(z_{it}) \right)$, therefore, $z_{it}^* = F^{-1} \left(1 - h_i A(i,t) / c_i^p \right)$. In addition, $\partial^2 TC / \partial z_{it}^2 = c_j^p C(i,t) f(z_{it}) = c_i^p C(i,t) f(z_{it})$, thus, $C(i,t) > 0$, then $\partial^2 TC / \partial z_{it}^2 > 0$, therefore, $z_{it}^* = F^{-1} \left(1 - h_i A(i,t) / c_i^p \right)$ is optimal. Taking z_{it}^* into $H_{itl} + P_{itl}$, it follows as Eq. (8). The solution procedure is shown in **Appendix B**.

$$H_{itl} + P_{itl} = h_i B(i,t) A(i,t) / 2 + c_i^p C(i,t) f(z_{it}^*) \tag{7}$$

3. Problem solving methodology

By modeling the problem, the optimal inventory factor z_{it}^* is mathematically converted into the variable depending on y_{itl}. The original problem variables are reduced to the decision variable x_{it}. The converted problem is similar to the dynamic deterministic lot-sizing with joint replenishment, except that the safety stock factor varies and the term $H_{itl} + P_{itl}$ with the replenishment cycle. Therefore, two typically effective heuristic algorithms in previous literature and three meta-heuristic algorithms for this problem are exerted. In addition, a novel modified heuristic is proposed.

3.1 Existing effective heuristic methods

Boctor et al. (2004) evaluated the outperformed performance of the FB-SK heuristic by Fogarty and Barringer (1987) and Silver E (1988). Robinson et al. (2007) proposed the Left-Right shift order heuristic with multiple items. These two methods are excellent, and it is necessary to adapt and apply them.

(1) FB-SK heuristic

First, the formulation of the model is modified to the dynamic programming model as $f_t = \min_{l \le t} \{ f_{l-1} + c_{lt} \}$, where, $f_1 = S_1 + \sum_{i=1}^{N} s_{i1}$ and $c_{lt} = S_l \delta_t + \sum_{i=1}^{N} \left(s_{il} x_{il} + H_{itl} + P_{itl} \right)$. f_t formulates the optimal costs for the interval of *t* periods. Using the DP model, calculate f_1. Then, calculate f_t for t=2, ..., T. Next, whether adding the order size of this item in a replenishment period to the previous order could save money is considered.

(2) Left-Right shift heuristic

Based on the research problem of this paper, the saving coefficient of Robinson' Left-Right shift heuristic is adapted. While the dealer *I'* order at period *t* is left shifted into period *t'*, it occurs that previous replenishment cycles $[t^-, t]$ and $[t, t^+]$ become replenishment cycles $[t^-, t']$ and $[t', t^+]$, respectively, where t^- denotes the last order period before t and t^+ denotes the first order period after *t*. This results in a change in the stock holding and the shortage cost for two replenishment cycles. The adapted saving coefficient $C_i(t, t')$ for Left-shifting item *i*'s order from period *t* into *t'* is Eq. (9), where $t^- \le t' \le t^+$. The adapted savings coefficient, $C(t, t')$, for rescheduling the family from *t* into *t'* is Eq.(10), where $\delta_{t'} = 1$ if it replenishes at period *t'* before shifting.

$$C_i\left(t,t'\right)=\left(x_{it'}-1\right)s_{it'}+s_{it}+\left(H_{it^-t}+P_{it^-t}+H_{itt^+}+P_{itt^+}\right)-\left(H_{it^-t'}+P_{it^-t'}+H_{it't^+}+P_{it't^+}\right) \quad (8)$$

$$C\left(t,t'\right)=\left(\delta_{t'}-1\right)S_{t'}+S_t-\sum_{i=1}^{N}C_i\left(t',t\right) \quad (9)$$

While partial order size for dealer *i* at period *t* is right shifted into period *t'*, it occurs that it replenishes between period *t* and t^+. The previous order cycle $[t, t^+]$ becomes two order cycles $[t, t']$ and $[t', t^+]$. This results in a change in the stock holding and the shortage cost for the time interval $[t, t^+]$. The adapted savings coefficient $C_i(t, t')$ for Right-shifting item *i*'s partial order size from period *t* into *t'* is Eq.(11), where $t \le t' \le t^+$ The adapted savings coefficient, $C(t', t)$, for rescheduling the family from *t* into *t'* is Eq. (12), where $\delta_{t'} = 1$ if an order is placed at period *t'* before shifting.

$$C_i\left(t,t'\right)=\left(x_{it'}-1\right)s_{it'}+s_{it}+\left(H_{itt^+}+P_{itt^+}\right)-\left(H_{itt'}+P_{itt'}+H_{it't^+}+P_{it't^+}\right) \quad (10)$$

$$C\left(t',t\right)=\left(\delta_{t'}-1\right)S_{t'}+S_t-\sum_{i=1}^{N}C_i\left(t',t\right) \quad (11)$$

3.2 Improved Left-Right shift heuristic

This paper improves the proposed Left-Right shift heuristic algorithm by Robinson, only considering individual cost savings and the family cost savings of shifting. We include additional cost savings with each positive cost savings for family shifting, which is, at each time *t*, all dealers whose $C_i(t, t')$ *is* positive are shifted, denoted by $C^+(t, t')$. Notably, multiple positive individual cost-saving shifts, which would have required multiple shifts, are simultaneously moved at once in the improved module. Thus, the improvement module accelerates the rate of decline of the objective function. The improved Left-Shift heuristic procedure is as follows. The pseudo-code is shown in Algorithm 1.

Algorithm 1 The improved Left-shift algorithm

```
1:  Input: horizon T; dealers I; demands u and standard deviation o; family setup cost s; individual setup cost s0; unit holding cost h; unit shortage
    penalty p; x_initial;
2:  Output: x_best and y_best;
3:  calculate matrix1, matrix2, and matrix3;
4:  while matrix2(t, t')>0 or matrix3(t, t')>0, ∃t, ∃t';
5:  % compute individual-shift cost-saving, update matrix1 (6th to 12th row);
6:      for i=1:I
7:        for t=first order period : last order period
8:          for t'=t⁻:t
9:            compute C_i(t, t'); matrix1(i, t')= C_i(t, t');
10:         end
11:       end
12:     end
13:     % compute positive family and family shift cost savings, update matrix2 and matrix3 (14th to 23th row);
14:     for t=1:T
15:         for t'=1:t
16:            for i=1:I
17:              if t⁻≤t', dealer i orders at t, and matrix1(i, t')>0
18:                a(i,t')=1;else a(i,t')=0;
19:              end
20:              if t⁻≤t' and dealer i orders at t
21:                β(i, t')=1;else β(i, t')=0;
22:              end
23:            end
24:            Compute Σ_{i∈I} α(i,t')*matrix1(i,t'); matrix2(t,t') = Σ_{i∈I} α(i,t)*matrix1(i,t');;
25:            Compute Σ_{i∈I} β(i,t')*matrix1(i,t'); matrix3(t,t') = Σ_{i∈I} β(i,t')*matrix1(i,t')
26:         end
27:     end
28:     select the max{matrix2(t, t'), matrix3(t, t')},all t, t';
29:     Left-shifting t to t' of dealers involving the maximum and generate x_new;
30:     compute y_new and if y_new <y and update x_best and y_best;
31: end
32: output x_best and y_best;
```

Step 1, generate an individual-shift cost-saving matrix, named *matrix1*, I×T dimension. For the initial solution, while it replenishes at t period and $t^{-}< t'<t$, calculate saving $C_i(t',t)$ for i, t', t and assign to *matrix1*(i, t'). Otherwise, $C_i(t',t)$ is assigned negative infinity. Step 2, generate a positive family-shift cost-saving matrix, *matrix2*, T×T dimension. For *matrix2*(t, t'), select *matrix1*(i, t') of all i while it replenishes at t period, $t^{-}< t'<t$, and *matrix1*(i, t'))>0. Furthermore, these selected *matrix1*(i, t') are accumulated and assigned to *matrix2*(t, t'). Step 3, generate family-shift cost-savings matrix, *matrix3*, T×T dimension. For matrix3(t, t'), select matrix1(i, t') of all i while an order is placed at t period, and $t^{-}< t'<t$. Furthermore, the selected matrix1(i, t') are accumulated and assigned to *matrix3*(t, t'). If *matrix2*$(t, t')<0$ and *matrix3*$(t, t')<0$, all t and t', stop. Step 4, Left-shifting. Select the maximum saving in *matrix1*, *matrix2*, and *matrix3*, Left-shifting the orders in t period into the period t', and update the solution. Return to Step 1.

The improved Right-shift heuristic is proposed. The pseudo-code is shown in Algorithm 2. Step 1, generate individual-shift cost-saving matrix, *matrix1*, I×T dimension. For the current solution, while an order of dealer i is placed at t period and $t< t'<t^{+}$, calculate $C_i(t',t)$ for $i, t,$ *and* t', and assign to *matrix1*(i, t'). Otherwise, $C_i(t',t)$ is assigned negative infinity. Step 2, generate positive family-shift cost-savings matrix, *matrix2*, T×T dimension. For *matrix2*(t,t'), select *matrix1*(i,t') of all i while it replenishes at t period, $t<t'<t^{+}$, and *matrix1*(i,t')>0. Furthermore, the selected *matrix1*(i, t') are accumulated and assigned to *matrix2*(t,t'). Step 3, generate family-shift cost-savings matrix, *matrix3*, T×T dimension. For *matrix3*(i, t'), select *matrix1*(i,t') of all i while it replenishes at t period and $t<t'<t^{+}$. Furthermore, the selected *matrix1*(i, t') are accumulated and assigned to *matrix3*(t,t'). If *matrix2*$(t, t')<0$ and *matrix3*$(t, t')<0$, all t and t', stop. Step 4, Right-shifting. Select the maximum saving in *matrix1*, *matrix2*, and *matrix 3*, place an order in the period t', and update the solution. Return to Step 1.

Algorithm 2 The improved Right-shift algorithm

1: **Input**: T; I; u; o; s; s0; h; p;
2: **Output**: x_{best} and y_{best};
3: Calculate *matrix1*, *matrix2*, and *matrix3*;
4: while matrix2$(t, t')>0$ or matrix3$(t, t')>0$, $\exists t, \exists t'$;
5: % compute individual shift cost savings, update *matrix1* (6th to 12th row);
6: For i=1:I
7: for t=first order period : last order period
8: for $t'=t:t^{+}$
9: compute $C_i(t\ t')$; *matrix1*$(i, t')=C_i(t,t')$;
10: end
11: end
12: end
13: % compute positive family and family shift cost savings, update *matrix2* and *matrix3* (14th to 23th row);
14: for t=1:T
15: for t'=t:T
16: for i=1:I
17: if $t^{+}\geq t'$, dealer i orders at t, and *matrix1*$(i,t')>0$
18: $a(i, t')$=1;else $a(i, t')$=0;
19: end
20: if $t^{+}\geq t'$ and dealer i orders at t
21: $\beta(i,t')$=1;else $\beta(i,t')$=0;
22: end
23: end
24: Compute $\sum_{i\in I}\alpha(i,t')*matrix1(i,t'); matrix2(t,t')=\sum_{i\in I}\alpha(i,t')*matrix1(i,t')$
25: Compute $\sum_{i\in I}\beta(i,t')*matrix1(i,t'); matrix3(t,t')=\sum_{i\in I}\beta(i,t')*matrix1(i,t')$
26: end
27: end
33: select the max{*matrix2*(t, t'), *matrix3*(t, t')},all t, t';
28: right shift t to t' of dealers involving the maximum and generate x_{new};
29: compute y_{new} and if $y_{new}<y$ and update x_{best} and y_{best};
30: end
31: output x_{new} and y_{new};

Note: t^{+} denotes the first ordered period after t.

4. Computational experiments and results discussion

In this section, five parameters, planning period set T∈{6,12,24}, number of dealers I∈{20,50,100,300}, unit period length L∈{7,15,30}, the minor setup cost, unit out-of-stock cost, are variously combined. Each dealer's order setup cost is randomly generated from 50 to 100 and multiplied by a variation factor set, {1,3,5}. The unit out-of-stock cost for each dealer is randomly generated from 500 to 1000 and multiplied by a variation factor set,{1,2,3}. Each dealer's unit stock-out cost is the unit inventory cost multiplied by a multiplier factor set, {30,60,90}. The other problem parameters are fixed. The unit demand u of each dealer is randomly drawn from 30 to 100, and the standard deviation of the distribution is its demand multiplied by a random factor between 0 and 1. The lead time of each period for each dealer is set to a random length within the cycle length. Thus, 324 data instances are generated based on the combination of the problem parameters. All instance experiments are conducted on a PC configured with an i7-10510U CPU, RAW 2.3Ghz, 16GB, and OS Windows 10. The data instances are tested by FB-SK, Left-shift heuristic, Right-shift heuristic, Left-Right shift heuristic, adaptive genetic algorithms (AGA), particle swarm algorithm (PSO), simulated annealing algorithm (SA), and improved Left-Right shift heuristic. The number of individuals in the population in AGA and PSO and the max iteration generations are set to 100 and 400, respectively. In PSO, the self-learning factor is 3, the inertia factor is 0.3, and the swarm-learning factor is 4. The dealer's replenishment decision for each period is represented as a gene position with 0 or 1, and the two-dimensional decisions table for all dealers during the planning horizon is the solution. In AGA, The selection operation adopts a roulette wheel strategy, the crossover strategy and mutation strategy adopt the single-point crossover the single-point mutation strategy, respectively.

Table 1 shows the performance indexes of 10 algorithms for 343 data instances. The FB-SK, LS, MLS2, LS-RS, and MLS-MRS heuristic algorithm exhibited superior performance in achieving the optimal objective value compared to the meta-heuristic algorithm, while exhibiting a significantly lower run time. The proposed MLS-MRS ranked first in *avg. opt.*, *avg. gap*, *max. gap* and std. dev. of gap, followed by LS-RS, and the heuristic is much less than the metaheuristic run time. Notably, the MLS-MRS time is 94.89% lower than the LS-RS. This confirms the time efficiency of the positive cost-saving family shifts module.

At *avg.opt*, MLS saves 3394 (ratio 0.13%) over LS. Although the percentage of optimized space is small, the runtime efficiency is improved by 90.36%. The runtime efficiency of the improved MLS is fully demonstrated. MLS-MRS is a sequential combination heuristic algorithm of MLS and MRS, which must run longer than MLS alone in terms of runtime, but with less optimal value savings. This indicates that MLS has optimized to a greater extent and the MRS optimization is less useful.

Table 1
Experimental results for algorithms

		avg. opt.	avg. opt. gap	max. opt. gap	std. dev. of opt. gap	avg. runtime
heuristic	FB-SK	2460939.02	0.63%	4.09%	0.0098	**0.42**
	LS	2457432.31	0.30%	3.01%	0.0057	104.46
	MLS	2454038.31	0.04%	1.16%	0.0014	10.07
	RS	2471967.46	0.75%	5.21%	0.0086	4.72
	MRS	2468812.28	0.69%	4.11%	0.0027	6.86
	LS_RS	2456744.25	0.25%	2.99%	0.0050	107.05
Metal heuristic	AGA	3815372.62	21.26%	93.64%	0.2401	273.14
	PSO	2482221.80	2.29%	20.47%	0.0386	709.80
	SA	2606577.78	1.54%	13.67%	0.0331	192.01
heuristic	MLS-MRS	**2453999.57**	**0.03%**	**1.05%**	**0.0013**	12.16

Table 2 shows the times both algorithms are simultaneously optimal in 324 instances, i.e., each cell is the times the corresponding algorithms in the row and column are simultaneously optimal. The cells on the diagonal are the times the algorithm is optimal. The last row of the table shows the times when each algorithm is uniquely optimal.The times of the optimal solution by the FB-SK, LS, MLS, LS-RS, AGA, PSO, and SA are 75, 188, 173, 203, 10, 31, 82, and 220, respectively.

Table 2
The number of inter-algorithm simultaneous optimality

	FB_SK	LS	MLS	RS	MRS	LS_RS	AGA	PSO	SA	MLS-MRS
FB-SK	**75**									
LS	75	**188**								
MLS	72	113	**173**							
RS	0	0	0	0						
MRS	0	0	0	0	0					
LS-RS	75	188	113	0	0	**203**				
AGA	9	9	9	0	0	9	**10**			
PSO	31	31	31	0	0	31	8	**31**		
SA	52	67	65	0	0	67	9	27	**82**	
MLS-MRS	72	113	173	0	0	113	9	31	65	**220**
	0	0	0	0	0	15	1	0	13	47

The last row shows the number of times when each algorithm is uniquely optimal.

The LS-RS is simultaneously optimal with the FB-SK 75 times and with the LS 188 times, reaching the times by the FB-SK and LS, respectively, whereas the times by the LS-RS are greater than 75 and 188. Therefore, the LS-RS is completely superior to the FB-Sk and LS. The MLS-MRS is simultaneously optimal with the MLS 173 times and with the PSO 31 times, reaching the times by MLS and PSO, whereas the times by the MLS-MRS are greater than 173 and 31. Therefore, MLS-MRS is completely superior to the MLS and PSO. The RS-LS and MLS-MRS have higher optimality times than the others, a total of 310 times (calculated as 220 plus 203 minus 113), accounting for 96.91%. The AGA and SA outperform all other algorithms by 1 and 13 times, respectively. The SA obtains the highest 82 times among the metaheuristic algorithms. However, 67 of 82 are simultaneously optimal with the LS and 65 with the MLS-MRS. This indicates that the SA outperforms other algorithms very rarely, remaining 13 times. Therefore, despite its 82 times in the metaheuristic algorithms, the SA does not perform well compared to the heuristic algorithms.

Two hundred twenty times by the MLS-MRS is the highest, while it is simultaneously optimal 72 times with the FB-SK, 113 times with the LS, 173 times with the MLS, 113 times with the LS-RS, 9 times with the AGA, 31 times with the PSO and 65 times with the SA. The difference in times between the MLS-MRS and the MLS is 47, calculated as 220 minus 173, which means that only the MLS step of the MLS-MRS obtains 173 times, and the MRS step obtains 47 times additionally. Once again, it proves that MLS has high optimization efficiency.

Given the above analysis, the LS-RS and MLS-MRS outperform the other algorithms. Of the 324 data instances, the LS-RS outperformed MLS-MRS 90 times with avg.opt. 3193539 and 3194001, respectively, and the opt. gap is 462 (0.014%). The optimality gap is very small. This indicates that the two algorithms are close to each other in these 90 times (calculated as 203 minus 113). On the contrary, the MLS-MRS outperformed LS-RS 107 times (calculated as 220 minus 113) , with avg.opt. 1396780 and 1405374, respectively, and the opt. gap is 8594 (0.61%). The latter optimality gap is larger. This indicates that the MLS-MRS outperforms the LS-RS in 107 times.

In summary, the MLS-MRS is superior to the LS-RS in terms of optimization performance and outperforms the other algorithms. In particular, the MLS-MRS significantly reduces the optimization time.

5. Conclusion and future research

This study analyzes the non-stationary stochastic lot-sizing with multi-dealer joint replenishments within an identical period. Additionally, there is no fill rate constraint for the problem, and a SDU policy is adopted. For the first time, we solve the problem of non-stationary stochastic LSP under the policy (R,S) and the multi-dealer joint replenishment. We propose the interval series for the planning horizon, considering the lead time of each period, which facilitates the formulation of the inventory and shortage costs. Thereby, the shortest path model is developed. By mathematically analyzing the inventory safety factors, its optimal value is expressed in terms of the replenishment decision. Thus, the model is transformed into a deterministic dynamic LSP. To solve the problem, the paper improves the very efficient and well-known Left-Right shift heuristic algorithm proposed by Robinson, which only considers individual cost savings and the family cost savings of shifting. We introduce an additional module, family shifting of positive cost savings, into Left-shifting and Right-shifting procedures, i.e., the individual positive cost-saving shifts are all operated at each iteration. The novel heuristic greatly accelerates the rate of decline in optimal value and improves the efficiency of the iterative search. This algorithm is compared with Robinson's LS-RS, AGA, PSO, SA, and the LS, RS, MLS, and MRS modules for 324 data instances.

The MLS-MRS and LS-RS significantly outperform the metaheuristic algorithm. In terms of the optimality gap, the MLS-MRS is 88% lower than the LS-RS. In terms of the maximum optimality gap, the MLS-MRS is 64.88% lower than the LS-RS. In terms of standard deviation, the MLS-MRS is 74% lower than the LS-RS. In terms of runtime, the MLS-MRS is 90.34% lower than the LS-RS. In terms of the optimal times, the MRS-MLS and LS-RS together account for 96.91%, which outperforms the other meta-heuristic algorithms. In the comparison of the optimal times between MRS-MLS and LS-RS, there are 113 simultaneous optimal times, and the remaining number of separate optimal times is 107 vs. 90. Although the difference in the times is not large, the average optimality gap ratio of the MLS-MRS is much larger than the LS-RS, 0.61%:0.014%. Overall, the MLS-LRS obtains the better optimum than LS-RS and other algorithms. More importantly, the time efficiency of the improved algorithm is greatly improved.

The study is a stochastic joint replenishment lot-sizing problem. The proposed model and algorithm can be directly used when the planning period is multi-period with non-stationary stochastic demand and joint replenishment of multiple items or dealers in management practice. Instead, it is not necessary to convert the stochastic demand to an expected value and then apply the deterministic lot-sizing mode. The static-dynamic uncertainty replenishment policy used in this paper is also recognized as a weak system tension, which facilitates the implementation of replenishment. In addition, the MLS-MRS algorithm is also applicable to the deterministic dynamic LSP. The MLS-MRS algorithm can be embedded into current supply chain replenishment intelligent decision-making systems as a deep algorithmic foundation.

With the study of this problem, it is possible to study further the integration and coordination between stochastic multi-project lot-sizing, location, or paths under the SDU policy, thus expanding the scope of the multi-problem non-stationary stochastic LSP.

CRediT authorship contribution statement

Jufeng Yang: Conceptualization, Methodology, Software, Validation, Writing-original draft, Visualization. Sujian Li: Conceptualization, Methodology, Writing-review &editing, Supervision.

Data availability

Data will be made available on request.

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Appendix A. The process of resolving $\partial TC/\partial z_{it}$.

If the order is placed for dealer *i* at the period *t*, e.g., $z_{it} > 0$

$$\partial TC/\partial z_{it} = \partial\left(\sum_{t=1}^{T}\left(S_t\delta_t + \sum_{i\in I} s_{it}x_{it}\right) + \sum_{i=1}^{N}\left(H_i^0 + P_i^0\right) + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{l=t}^{T}\left(s_{it}x_{it} + y_{itl}\left(H_{itl} + P_{itl}\right)\right)\right)\Big/\partial z_{it}$$

$$= \partial\left(h_i\left(B(i,t)/2 + z_{it}C(i,t)\right)A(i,t) + c_i^p C(i,t)\left(f\left(z_{it}\right) - z_{it}\left(1 - F\left(z_{it}\right)\right)\right)\right)\Big/\partial z_{it}$$

$$= h_i A(i,t)C(i,t) + c_i^p C(i,t)\,\partial\left(f\left(z_{it}\right) - z_{it}\left(1 - F\left(z_{it}\right)\right)\right)\Big/\partial z_{it}$$

$$= h_i A(i,t)C(i,t) - c_i^p C(i,t)\left(1 - F\left(z_{it}\right)\right)$$

where $\partial f\left(z_{it}\right)/\partial z_{it} = -z_{it} f\left(z_{it}\right)$, therefore, $\partial\left(f\left(z_{it}\right) - z_{it}\left(1 - F\left(z_{it}\right)\right)\right)\Big/\partial z_{it} = F\left(z_{it}\right) - 1$.

Appendix B. The function $H_{itl} + P_{itl}$ under the optimal z_{it}^* .

Take $z_{i\lambda}^*$ into $H_{itl} + P_{itl}$.

$$H_{itl} + P_{itl}$$

$$= h_i\left(B(i,t)/2 + z_{it}^* C(i,t)\right)A(i,t) + c_i^p C(i,t)\left(f\left(z_{it}^*\right) - z_{it}^*\left(1 - F\left(z_{it}^*\right)\right)\right)$$

$$= h_i\left(B(i,t)/2 + z_{it}^* C(i,t)\right)A(i,t) + c_i^p C(i,t)\left(f\left(z_{it}^*\right) - z_{it}^*\, A\left(i,t\right)h\big/c_i^p\right)$$

$$= h_i/2\, B(i,1)A(i,1) + C_i^p C(i,1) f\left(z_{i1}^*\right)$$

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