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Authorization or outsourcing? Investigating remanufacturing decisions under carbon trading policies and remanufacturing subsidies considering trade-in programs

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A B S T R A C T

With the introduction of carbon emission policies, subsidy policies, and the promotion of "trade-in" programs worldwide, determining the optimal remanufacturing strategy under various policy environments has become a critical issue. We develop six models to evaluate the effects of three policy combinations—carbon trading alone, carbon trading with consumer subsidies, and carbon trading with remanufacturer subsidies—under authorization and outsourcing remanufacturing strategies. The results show that dual policy of carbon emission trading and government subsidies more effectively promotes remanufacturing than a single carbon trading policy. When consumer subsidies reach a certain threshold, all supply chain members can achieve a win-win outcome, regardless of whether the remanufacturing strategy is authorization or outsourcing. The environmental cost is primarily influenced by carbon emissions from new products. If emissions are high, remanufacturer subsidies should be prioritized; if emissions are lower, consumer subsidies are more effective. Without subsidies, authorization has pricing advantages with low emissions, while outsourcing is more economical with high emissions or under market uncertainty. High carbon trading prices and subsidies increase overall supply chain profits but exhibit diminishing returns as excessive carbon prices increase corporate costs and reduce consumer surplus and social welfare. Moderate subsidies can mitigate these negative effects.

1. Introduction

Recently, due to technological advancements and accelerated consumer upgrades, the life cycle of high-tech products such as electronics and automobiles has significantly shortened, and technological iterations have become more frequent, leading to an exploding in the amount of garbage generated. For example, the collection and remanufacturing of batteries for electric vehicles have become increasingly critical issues as global new energy vehicles become more popular (Lin et al., 2023; Zhang et al., 2022, 2023a, 2023b). Companies like Tesla and BMW face the challenge of efficiently recycling and remanufacturing batteries, which involves not only effective resource utilization but also contributing to protecting the environment (Zhang et al., 2023b). In response, countries worldwide have introduced various emission reduction policies. For instance, the EU has started the "European Green Deal," which is designed to realize carbon neutrons by 2050; the U.S. government has passed the "Clean Air Act" to promote emission reductions in the power sector; and China, in its 14th Five-Year Plan, has set carbon discharges peak value and carbon neutrality targets while actively promoting the construction of a national carbon emission trading market (Cheng & Wang, 2023). Among these policies, carbon emission trading has become a preferred option for many countries due to its market flexibility and economic incentives for businesses (Chai et al., 2018; Cheng et al., 2022; Tsai et al., 2023; Yang et al., 2021; Zhu et al., 2024). The policy establishes an upper limit on total carbon footprints and permits companies to trade carbon emission quotas, incentivizing them to find more low-carbon production methods. In China, many companies have gradually shifted to low-carbon production methods by reducing carbon emission costs, making remanufacturing an important choice. However, determining the appropriate remanufacturing strategy in different policy environments has become a major challenge for companies.

The choice of remanufacturing strategy usually focuses on authorized remanufacturing or outsourced remanufacturing. Licensing remanufacturing refers to the practice where the original manufacturer authorizes a third-party enterprise to take

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remanufacturing, ensuring the quality and reliability of remanufactured products, which helps increase consumer trust and acceptance (Zhou et al., 2021). For example, Caterpillar Inc., a well-known global manufacturer in the field of construction machinery, authorizes third-party enterprises to take remanufacturing, resulting in high acceptance of its remanufactured goods in the market (Cheng et al., 2023). Apple grants Foxconn the rights to refurbish and remanufacture pre-owned iPhones, maintaining technical standards consistent with new products and preserving market reputation (Cao & Shao, 2024). Outsourcing remanufacturing, alternatively, involves the manufacturer outsourcing remanufacturing tasks to specialized remanufacturing firms, utilizing the technical expertise and experience of these firms to enhance the efficiency and quality of remanufacturing (Zhou et al., 2023). For example, Ford outsourced remanufacturing to third-party firms after failing in its remanufacturing attempts (Fang, 2023). HP chose to outsource its remanufacturing to third-party firms as well (Fang et al., 2023). Both strategies have pros and cons, and how firms make optimal decisions amid changes in policy environments is a key issue that needs further exploration.

Traditionally, the selection of remanufacturing strategies often considers only the choice between both newly produced and remanufactured items by initial consumers, ignoring the actual situation where replacement consumers purchase products through trade-in programs. In trade-in programs, consumers can receive discounts or subsidies by returning old products, which encourages the acquisition of either new or remanufactured items. The introduction of this factor makes remanufacturing strategies more complex and realistic. This study is the first to systematically incorporate trade-in programs into the discussion of remanufacturing strategy choices to explore optimal remanufacturing strategies under a more realistic market operation context. This approach not only addresses the existing research's neglect of replacement consumer behavior but also considers the combined effects of government policies (such as carbon emission trading (CET) policies and remanufacturing subsidy (RS) policies), providing more comprehensive decision-making guidance for companies and policymakers. Moreover, governments face challenges in policy selection. While the single CET policy has market-based advantages, its effectiveness may not match that of a dual policy combination in the absence of subsidies. Through the implementation of RS policies, governments can not only further incentivize companies to engage in remanufacturing but also enhance market acceptance of remanufactured products. The present literature mainly pays close attention to the effects of single policies such as no subsidies, consumer subsidies, and remanufacturer subsidies, with limited studies on policy combinations (Xu et al., 2024). Therefore, this study will deeply analyze how the combination of carbon emission trading and subsidy policies affects firms' decisions under different remanufacturing strategies, and how these policy combinations impact the overall progress of the remanufacturing. To sum up, we research the following questions:

(1) Should the government consider implementing RS policies in conjunction with the carbon emission trading policy and trade-in program?

(2) How do different forms of government remanufacturing subsidies influence the optimal remanufacturing strategy and pricing decisions of OEMs (Original Equipment Manufacturers)?

(3) What are the differences in the impact of single and dual government policies (carbon emission trading policy and government subsidy policy) on supply chain member profits, consumer surplus, social welfare, and environmental costs?

To solve these research questions, we consider a supply chain composed of a manufacturer (M), a remanufacturer (R), and a retailer (T). First, we construct six evaluation models under three government subsidy policies and two different remanufacturing strategies. Next, we compare and analyze product prices, supply chain member profits, consumer surplus, environmental effects, and social welfare in the absence of government subsidies, consumer subsidies, and RS policies in authorization and outsourcing remanufacturing strategies. We then explore the optimal remanufacturing strategy choice for OEMs under different combinations of RS policies (subsidizing remanufacturers vs. subsidizing consumers) and carbon emission policies. Finally, we analyze the effect of carbon trading prices and government subsidy amounts on total supply chain profits, consumer surplus, environmental impact, and social welfare. This paper is the first to systematically analyze the choice of remanufacturing strategies under the combined effects of carbon emission trading policies, remanufacturing subsidies, and trade-in programs, providing strategic guidance for firms in choosing remanufacturing strategies under different policies, and supporting governments in formulating optimal RS policies.

The key contributions of this thesis can be outlined as:

(1) This paper is the first to systematically incorporate the "trade-in" program into the discussion of remanufacturing strategy choices, considering the effect of replacement consumer behavior on remanufacturing strategy decisions. Previous research primarily focused on the initial consumer's choice between new and remanufactured products, overlooking the behavior of replacement consumers in trade-in programs. This research fills this void, enhancing its relevance to real-world market practices.

(2) This paper introduces the combined effects of government policies in the analysis of remanufacturing strategy choices. Existing research mainly discusses the effects of single policies (such as no subsidies, consumer subsidies, and remanufacturer subsidies) and rarely considers the impact of policy combinations. By systematically analyzing how the combination of carbon

emission trading policies and RS policies affects firms' remanufacturing strategy choices and pricing decisions, this paper fills a gap in current research.

(3) This paper provides policy implications for governments and strategic guidance for firms under different policy environments, aiding policymakers and companies in making better decisions to drive the progress of remanufacturing.

The structure of this paper is as follows: The literature review is placed in Section 2, while the model assumptions and demand functions are presented in Section 3. Then, in Section 4, we construct the theoretical model and solve for the equilibrium. Sections 5 and 6 provide comparative and simulation analyses. Section 7 discusses the conclusions and restrictions. See the Appendix for all proofs.

2. Literature review

This paper involves two main research themes: carbon emission control and subsidy policies, and sustainable operations management, which includes remanufacturing and trade-in programs.

2.1 Carbon emission control and subsidy policies

2.1.1 Carbon emissions trading policies

The Carbon Emission Trading (CET) policy has been proven to become an effective market mechanism that encourages companies to reduce carbon emissions by setting emission caps and allowing firms to trade surplus emission quotas. In recent years, numerous studies have examined the influences of CET policies on company behavior and environmental benefits. Liu et al. (2024) researched the operational mechanism of China's carbon market and highlighted its role in reducing emissions and promoting economic growth. Qi et al. (2023) researched the effect of CET policies on corporate competitiveness, finding that such policies can enhance environmental performance and market competitiveness. Pingkuo (2024) investigated the effect of CET on corporate production-making, revealing a tendency for corporations to use cleaner production technologies under a CET system. Tsai et al. (2023) incorporated carbon taxes and CET mechanisms into a green production mathematical programming model, analyzing the influences of different carbon cost structures (like carbon taxes, carbon caps, and trading) on corporate production structure and profitability. Xia et al. (2023) studied how manufacturers' choices regarding emission reduction affect authorized remanufacturing under CET frameworks and discovered that all three approaches to reducing emissions help to boost overall sales within the supply chain. Li et al. (2024) studied the best pricing strategies and emission reduction choices for OEMs and retailers across three remanufacturing frameworks, and they also analyzed how varying carbon tax and tariff levels influence OEMs' remanufacturing decisions, social welfare in importing nations, and environmental results.

Existing research mainly focuses on the effect of emissions trading policies on enterprise emission reduction and innovation, but few discuss the effect of the combination of emissions trading policies and RS policies. The purpose of this paper is to discuss the linkage influence of the simultaneous implementation of CET policy and RS policy and explore the comprehensive impact on the choice of remanufacturing model, firm pricing, social welfare, and environmental benefits.

2.1.2 Government subsidies

Subsidies from the government play a vital role in fostering sustainable development, particularly in encouraging green technological innovation and remanufacturing activities. Subsidy policies can lower production costs and strengthen the market competitiveness of remanufactured goods. For the past few years, increasingly, studies have focused on the influence of government subsidies in promoting environmental conservation and generating economic gains. Mitra and Webster (2008) investigated the influence of government subsidies (subsidizing remanufacturers, manufacturers, as well as subsidies that are shared between manufacturers and remanufacturers for remanufacturing. Zhang and Zhang (2022) examined the impact of government subsidies on the stability of the pricing system in authorized remanufacturing supply chains under the CET policy. Jiang (2023) established a game theory model composed of manufacturers (OEMs) and independent remanufacturers (IRs) to analyze the effects of carbon cap-and-trade mechanisms (CCT), government subsidies, and consumer education on production decisions. Chai et al. (2023) examined how various government subsidy policies (IR subsidy, retailer subsidy, and consumer subsidy (CS)) and carbon cap-and-trade policies (CTP) affect remanufacturing within a closed-loop supply chain composed of manufacturers, independent remanufacturers, and retailers. Dai et al. (2023) explored situations in which the government offers production or consumption subsidies to a supply chain involving manufacturers and remanufacturers, analyzing how these subsidies impact short-term competitive strategies and long-term behavioral evolution among supply chain participants. Xu et al. (2024) considered scenarios with no subsidy, consumer subsidy, and corporate subsidy, analyzing the impact of dual government policies (including CCT and subsidies) on remanufacturing.

While these studies have well-explored remanufacturing and production decisions under single or dual policy scenarios, they have not examined the impact of different subsidy policies (such as subsidies for remanufacturers and consumers) in conjunction with CET policies on the choice of remanufacturing modes. This study aims to systematically explore the synergy between CET policies and RS policies, analyzing the optimal choice of remanufacturing modes under different subsidy strategies and dual policy scenarios.

2.2 Sustainable operations management

2.2.1 Trade-in programs

Trade-in programs are an important strategy for promoting the recovery and remanufacturing of scrapped items. By encouraging consumers to trade in their old ones, companies can ensure a continuous supply of used products, thereby supporting the sustainable development of remanufacturing businesses. Cao et al. (2022) constructed a mathematical model on account of two trade-in refund policies (full refund and partial refund) to explore the optimal strategy choices, finding that equilibrium strategies depend on consumer satisfaction with new products and the residual value of the product. Bai et al. (2023) constructed a game theory model to explore how firms should adjust their trade-in strategies (TU and TUC modes) in response to the appearance of third-party resale platforms. Liu et al. (2024) considered a scenario where manufacturers authorize a third-party information platform (3IP) to handle trade-ins and compared three models: no trade-in strategy, trade-in through 3IP, and trade-in through wholesale contracts, finding that the manufacturer gains more profit when authorizing 3IP. Schepler et al. (2023) used mixed-integer linear programming to model trade-in, processing, and resale processes related to second-hand consumer electronics, proposing a heuristic approach based on mixed-integer programming to provide managerial insights. Tang et al. (2023) constructed a dual-head monopoly model to analyze the influence of brand loyalty on exclusive and non-exclusive trade-in programs, discovering that non-exclusive trade-in programs are better for both companies when brand loyalty is sufficiently strong. Li et al. (2023) developed an infinite-horizon model to investigate the conditions under which companies choose to adopt trade-in programs in the appearance of P2P second-hand markets and determine the prices of new products with and without a trade-in.

Existing literature primarily examines the implementation effects of trade-in programs and their impact on the market, but there is a lack of studies exploring the synergy between trade-in programs, CET policies, and RS policies. This study aims to analyze trade-ins and remanufacturing under the background of dual policies, incorporating trade-in programs into the discussion of remanufacturing mode choices to explore the optimal remanufacturing mode in scenarios more aligned with actual market practices.

2.2.2 Remanufacturing

As a circular economy model, remanufacturing can extend the product life cycle, decrease resource usage, and lessen environmental pollution. In recent years, researchers have extensively studied different remanufacturing modes and their advantages from both an economic and ecological perspective. Guo et al. (2024) examined the impact of CET on the manufacturing/remanufacturing supply chain by constructing a game model of OEMs and outsourced remanufacturers (ORs) under decentralized and centralized decision conditions. Xia et al. (2023) also examined how three different emission reduction strategies — OEMs independently reducing emissions, authorized remanufacturers independently reducing emissions, and a collaborative approach to emission reduction—affect outsourced remanufacturing. Li et al. (2024) developed a remanufacturing model considering carbon taxes and trade-in programs, finding that the optimal decision factors differ between emerging and mature market scenarios. Zhu et al. (2022) explored dynamic production and CER adjustment tactics for both new and remanufactured products under mixed carbon regulatory frameworks (carbon taxes and cap-and-trade systems), considering bounded rationality, exploring how adjustment parameters and critical factors affect key decision factors, anticipated profits, and overall carbon emissions. Zhu et al. (2024) incorporated green technology, remanufacturing, consumer environmental awareness, and hybrid carbon policies (carbon tax + cap-and-trade) into a dynamic framework using differential game theory, examining the optimal decisions under four scenarios: idealized, non-cooperative, single-party, and dual-party cost-sharing agreements.

Most existing studies focus on remanufacturing modes and their economic benefits but seldom combine CET policies and RS policies to explore their combined effects. This study is the first to systematically incorporate trade-in programs into the discussion of remanufacturing mode choices, exploring the optimal remanufacturing mode in a context more aligned with actual market operations. This not only addresses the gap in existing research regarding replacement consumer behavior but also considers the combined effects of government policies, such as CET and RS policies, providing more comprehensive decision-making guidance for firms and policymakers.

3. Model description and assumptions

3.1 Problem description

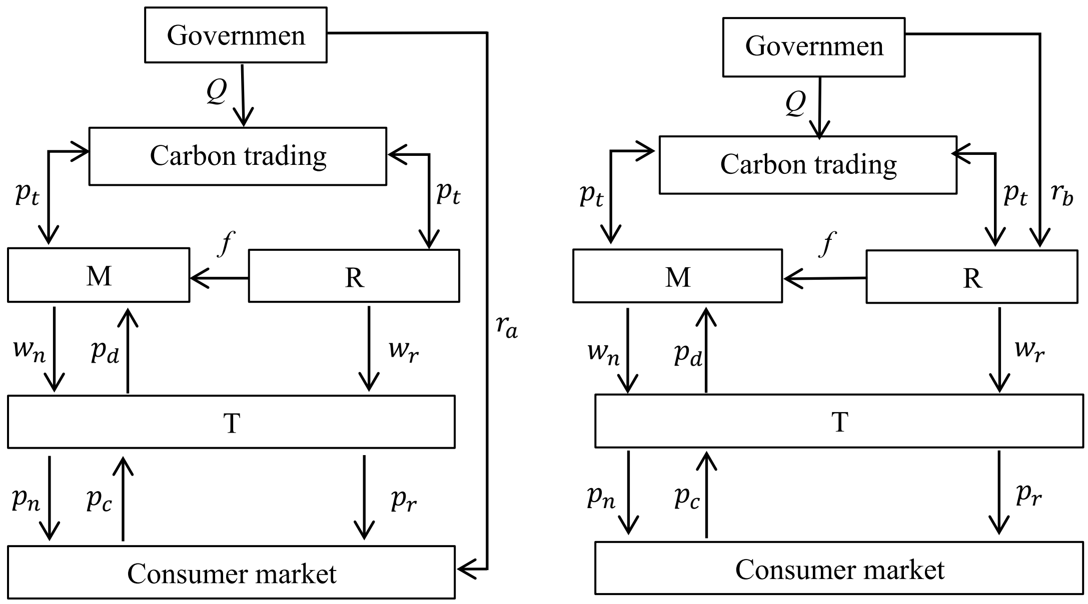
To achieve the "dual carbon goals," governments worldwide have enacted various policies to incentivize corporate carbon emission reductions. Among these, CET and government subsidies are crucial market-driven and economic incentive policies. Remanufacturing and trade-in programs are essential means for enterprises to enhance environmental sustainability. Therefore, we analyze a one-period supply chain made up of an original equipment manufacturer (OEM, M) a remanufacturer

(R), and a retailer (T). We construct a three-stage game model under different government subsidy strategies to achieve the dual carbon goals involving M, R, and T. The decision-making order for the three-stage game model is as follows:

1. In the first stage, the government decides on the subsidy strategy: (a) only implementing the emission trading policy, (b) implementing both the emission trading and CS policies and (c) implementing both the CET and RS policies.

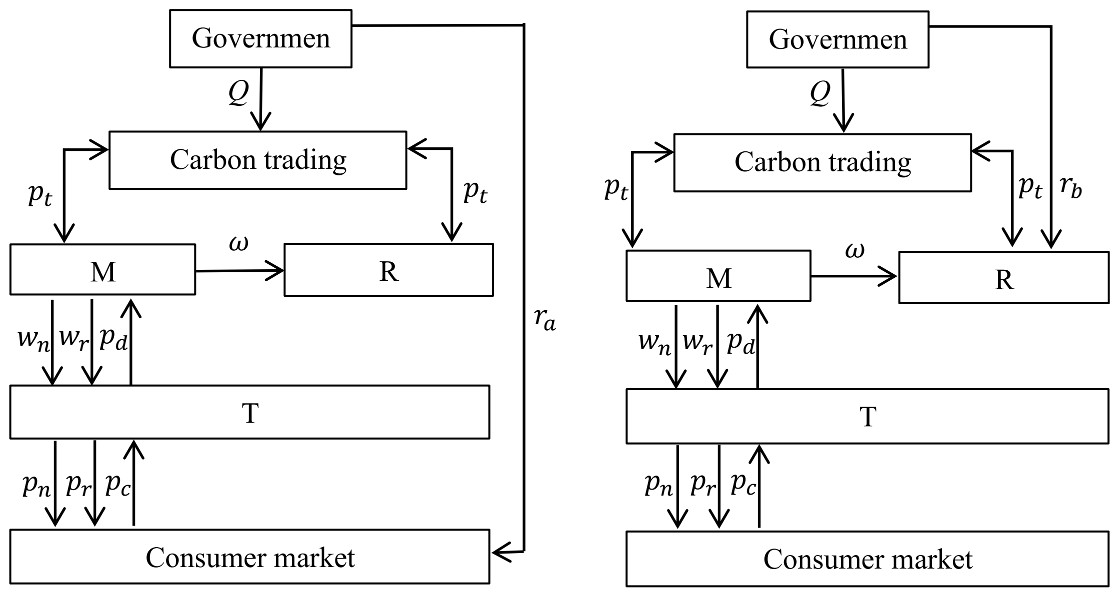
2. Manufacturers decide on the remanufacturing method and choose to authorize or outsource remanufacturing. For the authorized mode, the manufacturer determines the authorization fee, sets the wholesale price for new items, and defines the trade-in discount, while the remanufacturer establishes the wholesale price for remanufactured goods. In contrast, if the outsourced mode is chosen, the remanufacturer sets the outsourcing fee, and the manufacturer decides on the wholesale prices of new and remanufactured products, along with the retailer's trade-in discount.

3. In the third stage, the retailer decides on the retail prices of new and remanufactured products and the trade-in discount offered to consumers. Fig. 1 and Fig. 2 illustrate the model framework and decision-making aspects.



(a) Consumer subsidy (b) Remanufacturer subsidy

Fig. 1. Authorization remanufacturing



(a) Consumer (b) Remanufacturer subsidy

Fig. 2. Outsourcing remanufacturing

3.2 Model assumptions

To further study the impact of CET and trade-in programs on the optimal remanufacturing mode choice and pricing decisions of OEMs under the dual carbon goals and to analyze the optimal choice of remanufacturing subsidies, the following assumptions are proposed:

Assumption 1. Drawing on the existing literature (Fan et al., 2022), the production costs per unit for new and remanufactured products are labeled as c_n and $c_n - \Delta$, respectively, with Δ indicating the cost savings per remanufactured unit.

Assumption 2. Based on prior research (Yi et al., 2022), e represents carbon emissions per unit for new products, whereas $e - E$ represents carbon emissions per unit for remanufactured products, where E stands for the carbon emission savings per remanufactured unit, satisfying $E < e$.

Assumption 3. Drawing on the research of Guo et al. (2022), θ denotes consumer willingness to purchase new products, uniformly distributed between [0,1]. $\delta\theta$ indicates the consumer's propensity to buy remanufactured items, while δ signifies the acceptance level for these remanufactured goods, falling within the range of 0 to 1.

Assumption 4. Drawing on the research of Hu et al. (2023), based on Assumption 3, the utility functions for consumers purchasing new and remanufactured products are the same under the no subsidy policy and the RS policy, i.e. $u_{nn} = \theta - p_n$ and $u_{nr} = \beta\theta - p_r$. The utility functions for consumers purchasing new and remanufactured products under the CS policy are different. The utility functions for participating in the trade-in program and continuing to hold old products are also provided. Under a CS policy, the utility functions for consumers purchasing new and remanufactured products are defined as $u_{nn} = \theta - p_n$ and $u_{nr} = \beta\theta + r_a - p_r$, respectively. The utility function for participating in a trade-in program is $u_{on} = (1-\delta)\theta - p_n + p_d$, while the utility function for retaining the current product is $u_{oo} = \delta\theta$.

Table 1
Symbols and Their Meanings

Symbols	Definition
$j = OEM, CM, R$	Nature of the firm: M for manufacturer, R for remanufacturer, T for retailer
$i = n, r$	Product type: n for new product, r for remanufactured product
θ	Willingness to purchase new products, $0 < \theta < 1$
α	Proportion of new consumers in the market
δ	Durability parameter of old products, $0 < \delta < 1$
β	Acceptance of remanufactured products, $0 < \delta < \beta < 1$
v	Unit residual value of old products
r_a	Government subsidy for consumers purchasing remanufactured products
r_b	Government subsidy for remanufacturers producing remanufactured products
p_t	CET price
w_i^k	Wholesale price of product i under mode k
p_i^k	Retail price of product i under mode k
p_c^k	Trade-in rebate paid by the manufacturer to the retailer under mode k
p_d^k	Trade-in discount offered by the retailer to consumers under mode k
f	Authorization fee per unit remanufactured product
ω	Outsourcing fee per unit remanufactured product
d_{nn}^k	Demand for new products under mode k
d_{nr}^k	Demand for remanufactured products under mode k
d_{on}^k	Demand for trade-in products under mode k
π_j^k	Profit of firm j under mode k

Assumption 5. Based on the above utility functions, consumers will opt to purchase new products only when $u_{nn} \geq \max\{0, u_{nr}\}$. Likewise, they will select remanufactured products only when $u_{nr} \geq \max\{0, u_{nn}\}$. Consequently, in scenarios involving no subsidy or a RS policy, the demand functions for new and remanufactured products are represented by $d_{nn} = \alpha \int_{\frac{p_n - p_r}{1-\beta}}^{1} d\theta = \alpha(1 - \frac{p_n - p_r}{1-\beta})$ and $d_{nr} = \alpha \int_{\frac{p_r}{\beta}}^{\frac{p_n - p_r}{1-\beta}} d\theta = \alpha(\frac{p_n - p_r}{1-\beta} - \frac{p_r}{\beta})$, respectively. The demand function for participating in a trade-in program is $d_{on} = (1-\alpha)\int_{\frac{p_n - p_d}{1-2\delta}}^{1} d\theta = (1-\alpha)(1 - \frac{p_n - p_d}{1-2\delta})$.

Assumption 6. According to Xu et al. (2024), the government allocates a carbon emission quota , denoted as Q, to enterprises. If the manufacturer's carbon emissions exceed (or are below) this quota, the manufacturer buys (sells) carbon credits in the carbon market. The CET price p_t is a linear function of the carbon quota, $p_t = T - \gamma Q$, where T is a constant and γ represents the elasticity coefficient of the carbon quota on the CET price.

For readability, Table 1 sums up the relevant symbols.

4. Model construction and solution

To analyze how carbon emission regulation and subsidy policies affect the optimal choice of remanufacturing strategy and pricing decisions for manufacturers, along with their impact on the environment, consumer surplus, and social welfare, we initially developed a game model under the authorized remanufacturing mode, with scenarios of no government subsidy, CS, and RS, and derived an equilibrium solution in Section 4.1. Subsequently, in Section 4.2, we formulated the game model under the outsourcing remanufacturing mode, including scenarios of no government subsidy, CS, and RS, and obtained the corresponding equilibrium outcomes.

4.1 Authorized remanufacturing mode

4.1.1 No government subsidy

Under the authorized remanufacturing mode with no government subsidy (Model AW), the manufacturer (M) is responsible for producing and wholesaling new products to the retailer and recovering old products through trade-in programs. M authorizes the remanufacturer (R) to produce and sell remanufactured products. M profits from new product sales and authorization fees collected from R. R pays an authorization fee to M for each remanufactured product sold and profits from remanufactured product sales. Finally, the retailer (T) sells new and remanufactured products to the market and recovers old products through trade-in programs. In Model AW, the formulas for calculating the profits of M, R, and T are as outlined below:

$$\pi_M^{AW} = (w_n - c_n)d_{nn}^{AW} + (w_n - c_n - p_c + v)d_{on}^{AW} - p_t(e(d_{nn}^{AW} + d_{on}^{AW}) - Q) + f d_{nr}^{AW};$$

$$\pi_R^{AW} = (w_r - c_n + \Delta - f)d_{nr}^{AW} - p_t((e - E)d_{nr}^{AW} - Q);$$

$$\pi_T^{AW} = (p_n - w_n)d_{nn}^{AW} + (p_n - w_n + p_c - p_d)d_{on}^{AW} + (p_r - w_r)d_{nr}^{AW}.$$

The order of decision-making in the game proceeds as follows: Initially, the manufacturer decides the authorization fee (f). Next, the manufacturer sets the wholesale price of new products (w_n) and the trade-in incentive for the retailer (p_c), while the remanufacturer establishes the wholesale price for the remanufactured goods (w_r). Ultimately, the retailer determines the retail pricing for both new (p_n) and remanufactured products (p_r), as well as the trade-in discount offered to consumers (p_d). We then derive the equilibrium results for Model AW using backward induction, which are detailed in Lemma 1.

Lemma 1. In Model AW, the manufacturer's most advantageous pricing strategies and resultant profits are delineated as follows:

$$f = \frac{(-ep_t - c_n + 1)\beta^2 + 8(E-e)p_t - 8(c_n - \delta - \beta)}{2(\beta+8)},$$

$$w_n = \frac{(3ep_t + 3c_n + 1)\beta + 2(E+3e)p_t + 2(\Delta + 3c_n + 4)}{2(\beta+8)},$$

$$p_c = \frac{(2ep_t + 2\delta + v + 2c_n)\beta + 2(E-e)p_t + 8(v + 2\delta) + 2(\Delta - c_n)}{2(\beta+8)},$$

$$\pi_M^{AW} = \frac{\alpha((3ep_t + c_n + 1)\beta + (2E+6e)p_t + 2\Delta - 10c_n + 8)(-\beta^2(ep_t + c_n - 1) - \beta(ep_t + c_n - 7) + 2p_t(3E+e) + 6\Delta + 2c_n - 8)}{8(\beta+8)^2(\beta-1)} +$$
$$\frac{(\alpha-1)(-ep_t + 2\delta - v + c_n - 1)(ep_t + 2\delta - v + c_n - 1)}{8(2\delta-1)} -$$

$$p_t\left(e\left(\frac{\alpha(-\beta^2(ep_t + c_n - 1) - \beta(ep_t + c_n - 7) + 2p_t(3E+e) + 6\Delta + 2c_n - 8)}{4(\beta+8)(\beta-1)} - \frac{(\alpha-1)(ep_t + 2\delta - v + c_n - 1)}{4(2\delta-1)}\right) - Q\right) -$$

$$\frac{(\beta+2)\left((-ep_t - c_n + 1)\beta^2 + 8(E-e)p_t - 8c_n + 8\Delta + 8\beta\right)\alpha((ep_t + c_n)\beta + (E-e)p_t + \Delta - c_n)}{4(\beta+8)^2(\beta-1)\beta};$$

the remanufacturer's most favorable pricing strategies and corresponding profits are as detailed below:

$$w_r = \frac{(ep_t + c_n + 1)\beta^2 + 4(ep_t + c_n + 2)\beta + 4(e-E)p_t + 4(c_n - \Delta)}{2(\beta+8)},$$

$$\pi_R^{AW} = \frac{-4\left(\frac{1}{2}(ep_t + c_n + 1)\beta^2 + \left(\left(E + \frac{e}{2}\right)p_t + \frac{c_n}{2} - \frac{v}{4} + \Delta - \frac{3\delta}{2} + 4\right)\beta + \frac{11}{2}(E-e)p_t - \frac{11c_n}{2} - 2v + \frac{11\Delta}{2} - 12\delta\right)((ep_t + c_n)\beta + (E-e)p_t + \Delta - c_n)(\beta+2)\alpha + 8Q\beta p_t(\beta-1)(\beta+8)^2}{8(\beta+8)^2(\beta-1)\beta};$$

the retailer's most favorable pricing strategies and corresponding profits are as detailed below:

$$p_n = \frac{3(ep_t+c_n+1)\beta+2(E+3e)p_t+2(\Delta+3c_n+12)}{4(\beta+8)},$$

$$p_r = \frac{(ep_t+c_n+3)\beta^2+4(ep_t+c_n+6)\beta+4(e-E)p_t+4(c_n-\Delta)}{4(\beta+8)},$$

$$p_d = \frac{(2ep_t+6\delta+v+2c_n)\beta+2(E-e)p_t+8v+2\Delta+48\delta-2c_n}{4(\beta+8)},$$

$$\pi_T^{AW} = \frac{(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+2p_t(3E+e)+6\Delta+2c_n-8)\left((-3ep_t-3c_n+1)\beta-2(E+3e)p_t-2\Delta-6c_n+8\right)\alpha}{16(\beta+8)^2(\beta-1)} - \frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)(v-c_n-2\delta-ep_t)}{16(2\delta-1)} - \frac{(\beta+2)\left((-ep_t-c_n+1)\beta^2+4(-ep_t-c_n+2)\beta+4(E-e)p_t+4(\Delta-c_n)\right)\alpha\left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n\right)}{8(\beta+8)^2(\beta-1)\beta};$$

the needs of new and old customers are:

$$d_{nn} = \frac{\alpha(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+2p_t(3E+e)+6\Delta+2c_n-8)}{4(\beta+8)(\beta-1)},$$

$$d_{nr} = \frac{\alpha\left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n\right)(\beta+2)}{2(\beta+8)(1-\beta)\beta},$$

$$d_{on} = \frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)}{4(1-2\delta)}.$$

4.1.2 Consumer subsidy

In the authorized remanufacturing mode with a CS policy (Model AC), the formulas for calculating the profits of M, R, and T are as outlined below:

$$\pi_M^{AC} = (w_n-c_n)d_{nn}^{AC}+(w_n-c_n-p_c+v)d_{on}^{AC}-p_t(e(d_{nn}^{AC}+d_{on}^{AC})-Q)+fd_{nr}^{AC};$$

$$\pi_R^{AC} = (w_r-c_n+\Delta-f)d_{nr}^{AC}-p_t((e-E)d_{nr}^{AC}-Q);$$

$$\pi_T^{AC} = (p_n-w_n)d_{nn}^{AC}+(p_n-w_n+p_c-p_d)d_{on}^{AC}+(p_r-w_r)d_{nr}^{AC}.$$

The game's sequence follows the same pattern as described in Section 4.1.1. Using backward induction, we derive the equilibrium results for Model AC, which are detailed in Lemma 2.

Lemma 2. In Model AC, the manufacturer's most advantageous pricing strategies and resultant profits are delineated as follows:

$$f = \frac{\beta^3+\left((E-5e)p_t+\Delta-5c_n+7\right)\beta^2+\left((3E+e)p_t+3\Delta+c_n-5r_a-8\right)\beta+4(e-E)p_t+4(c_n-r_a-\Delta)}{5\beta^2-\beta-4},$$

$$w_n = \frac{7\beta^2+(2Ep_t+2\Delta-5r_a-5)\beta-2(Ep_t+\Delta+2r_a+1)}{5\beta^2-\beta-4},$$

$$p_c = \frac{(-5(ep_t-v+c_n)+14)\beta^2+\left((4E+e)p_t-v+4\Delta+c_n-10r_a-10\right)\beta+4(e-E)p_t+4(-v-\Delta+c_n-2r_a-1)}{2(5\beta^2-\beta-4)},$$

$$\pi_M^{AC} = \frac{\alpha\left((-5c_n+7)\beta^2+(2Ep_t+2\Delta+c_n-5r_a-5)\beta-2Ep_t-2\Delta+4c_n-4r_a-2\right)(\beta+2)(Ep_t+\Delta+\beta-1)}{2(5\beta^2-\beta-4)^2}+\frac{(ep_t-v+c_n)(ep_t+v-c_n)(\alpha-1)}{8(2\delta-1)} - p_t\left(e\left(\frac{\alpha(\beta+2)(Ep_t+\Delta+\beta-1)}{2(5\beta^2-\beta-4)}+\frac{(\alpha-1)(ep_t-v+c_n)}{4(2\delta-1)}\right)-Q\right) - \frac{3\alpha\left(\beta^3+\left((E-5e)p_t+\Delta-5c_n+7\right)\beta^2+\left((3E+e)p_t+3\Delta+c_n-5r_a-8\right)\beta+4(e-E)p_t+4(c_n-r_a-\Delta)\right)(Ep_t+\Delta+\beta-1)}{2(5\beta^2-\beta-4)^2};$$

the remanufacturer 's most favorable pricing strategies and corresponding profits are as detailed below:

$$w_r = \frac{4\beta^3+(-Ep_t-\Delta+1)\beta^2+(Ep_t+\Delta-5r_a-5)\beta-4r_a}{5\beta^2-\beta-4},$$

$$\pi_R^{AC} = \frac{(50Qp_t-9\alpha)\beta^3+\left((-18Ep_t-18\Delta+18)\alpha+30p_tQ\right)\beta^2+\left(-9\alpha(Ep_t+\Delta-1)^2-48p_tQ\right)\beta-32p_tQ}{50\beta^3+30\beta^2-48\beta-32};$$

the retailer's most favorable pricing strategies and corresponding profits are as detailed below:

$$p_n = \frac{6\beta^2+(Ep_t+\Delta-5r_a-3)\beta-Ep_t-\Delta-4r_a-3}{5\beta^2-\beta-4},$$

$$p_r = \frac{9\beta^3+(-Ep_t-\Delta)\beta^2+(Ep_t+\Delta-10r_a-9)\beta-8r_a}{2(5\beta^2-\beta-4)},$$

$$p_d = \frac{(-5(ep_t-v+c_n)+24)\beta^2+\left((4E+e)p_t-v+4\Delta+c_n-20r_a-12\right)\beta+4(e-E)p_t+4(-v-\Delta+c_n-4r_a-3)}{4(5\beta^2-\beta-4)},$$

$$\pi_T^{AC} = \frac{-\alpha(\beta+2)(Ep_t+\Delta+\beta-1)^2}{2(5\beta+4)(5\beta^2-\beta-4)} + \frac{(1-\alpha)(v-ep_t-c_n)^2}{8(2\delta-1)} - \frac{3\alpha\beta(Ep_t+\Delta+\beta-1)^2}{4(5\beta+4)(5\beta^2-\beta-4)};$$

the needs of new and old customers are:

$$d_{nn} = \frac{\alpha(\beta+2)(Ep_t+\Delta+\beta-1)}{2(5\beta+4)(\beta-1)},$$

$$d_{nr} = \frac{3(Ep_t+\Delta+\beta-1)\alpha}{2(5\beta+4)(1-\beta)},$$

$$d_{on} = \frac{(1-\alpha)(ep_t-v+c_n)}{4(1-2\delta)}.$$

4.1.3 Remanufacturer subsidy

In the authorized remanufacturing mode with a RS policy (Model AR), the formulas for calculating the profits of M, R, and T are as outlined below:

$$\pi_M^{AR} = (w_n-c_n)d_{nn}^{AR} + (w_n-c_n-p_c+v)d_{on}^{AR} - p_t(e(d_{nn}^{AR}+d_{on}^{AR})-Q) + fd_{nr}^{AR};$$

$$\pi_R^{AR} = (w_r+r_b-c_n+\Delta-f)d_{nr}^{AR} - p_t((e-E)d_{nr}^{AR}-Q);$$

$$\pi_T^{AR} = (p_n-w_n)d_{nn}^{AR} + (p_n-w_n+p_c-p_d)d_{on}^{AR} + (p_r-w_r)d_{nr}^{AR}.$$

The game's sequence follows the same pattern as described in Section 4.1.1. Using backward induction, we derive the equilibrium results for Model AR, which are detailed in Lemma 3.

Lemma 3. In Model AR, the manufacturer's most advantageous pricing strategies and resultant profits are delineated as follows:

$$f = \frac{(-ep_t-c_n+1)\beta^2+8\beta+8(E-e)p_t+8(\Delta-c_n+r_b)}{2(\beta+8)},$$

$$w_n = \frac{(3ep_t+3c_n+1)\beta+2(E+3e)p_t+2\Delta+6c_n+2r_b+8}{2(\beta+8)},$$

$$p_c = \frac{(2ep_t+2\delta+v+2c_n)\beta+2(E-e)p_t+2r_b+8v+2\Delta+16\delta-2c_n}{2(\beta+8)},$$

$$\pi_M^{AR} = \frac{(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+6(Ep_t+\Delta+r_b)+2ep_t+2c_n-8)\left((3ep_t+c_n+1)\beta+2(E+3e)p_t+2\Delta-10c_n+2r_b+8\right)\alpha}{8(\beta-1)(\beta+8)^2} + \frac{(ep_t+v+1-c_n-2\delta)^2}{4} - p_t\left(e\left(\frac{(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+6(Ep_t+\Delta+r_b)+2ep_t+2c_n-8)\alpha}{4(\beta+8)(\beta-1)} - \frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)}{4(2\delta-1)}\right)-Q\right) - \frac{\left((-ep_t-c_n+1)\beta^2+8(E-e)p_t+8(\Delta+\beta-c_n+r_b)\right)\alpha(\beta+2)((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n+r_b)}{4(\beta+8)^2(\beta-1)\beta};$$

the remanufacturer's most favorable pricing strategies and corresponding profits are as detailed below:

$$w_r = \frac{(ep_t+c_n+1)\beta^2+4(ep_t+c_n+2)\beta+4(e-E)p_t-4(\Delta-c_n+r_b)}{2(\beta+8)},$$

$$\pi_R^{AR} = \frac{-\alpha\left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n+r_b\right)^2(\delta+2)^2+2Q\beta p_t(\beta-1)(\beta+8)^2}{2(\beta+8)^2(\beta-1)\beta};$$

the retailer's most favorable pricing strategies and corresponding profits are as detailed below:

$$p_n = \frac{3(ep_t+c_n+1)\beta+2(E+3e)p_t+2\Delta+6c_n+2r_b+24}{4(\beta+8)},$$

$$p_r = \frac{(ep_t+c_n+3)\beta^2+4(ep_t+c_n+6)\beta+4(e-E)p_t-4(\Delta-c_n+r_b)}{4(\beta+8)},$$

$$p_d = \frac{(2ep_t+6\delta+v+2c_n)\delta+2(E-e)p_t+2r_b+8v+2\Delta+48\delta-2c_n}{4(\beta+8)},$$

$$\pi_T^{AR} = \frac{((-3ep_t-3c_n+1)\beta-2(E+3e)p_t-2\Delta-6c_n-2r_b+8)\alpha(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+6(Ep_t+\Delta+r_b)+2ep_t+2c_n-8)}{16(\beta+8)^2(\beta-1)} - \frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)(v+1-ep_t-c_n-2\delta)}{16(2\delta-1)} - \frac{(((-ep_t-c_n+1)\beta^2-4(ep_t+c_n-2)\beta+4(E-e)p_t+4(\Delta-c_n+r_b))\alpha(\beta+2))((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n+r_b)}{8(\beta+8)^2(\beta-1)\beta};$$

the needs of new and old customers are:

$$d_{nn} = \frac{\alpha(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+6(Ep_t+r_b+\Delta)+2(ep_t+c_n-4))}{4(\beta+8)(\beta-1)},$$

$$d_{nr} = \frac{\alpha(\beta+2)((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n+r_b)}{2(\beta+8)(1-\beta)\beta},$$

$$d_{on} = \frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)}{4(1-2\delta)}.$$

4.2 Outsourced remanufacturing mode

4.2.1 No government subsidy

Under the outsourced remanufacturing mode with no government subsidy (Model OW), the manufacturer (M) is responsible for producing and wholesaling new products to the retailer and recovering old products through trade-in programs. After paying an outsourcing fee to the remanufacturer (R), M delegates the production of remanufactured products to R. M obtains new and remanufactured products from R through wholesale price contracts and then wholesales them to the retailer (T). Finally, T sells new and remanufactured products to the market and recovers old products through trade-in programs. In Model OW, the formulas for calculating the profits of M, R, and T are as outlined below:

$$\pi_M^{OW} = (w_n - c_n)d_{nn}^{OW} + (w_n - c_n - p_c + v)d_{on}^{OW} + (w_r - \omega)d_{nr}^{OW} - p_t(e(d_{nn}^{OW} + d_{on}^{OW}) - Q);$$

$$\pi_R^{OW} = (\omega - c_n + \Delta)d_{nr}^{OW} - p_t((e - E)d_{nr}^{OW} - Q);$$

$$\pi_T^{OW} = (p_n - w_n)d_{nn}^{OW} + (p_n - w_n + p_c - p_d)d_{on}^{OW} + (p_r - w_r)d_{nr}^{OW}.$$

In the case of outsourcing, the game sequence begins with the manufacturer determining the outsourcing fee (ω). Then, the manufacturer sets the wholesale prices for both new and remanufactured products (w_n and w_r), along with the trade-in rebate to the retailer (p_c). Finally, the retailer determines the retail prices of new and remanufactured products (p_n and p_r) and the trade-in discount to consumers (p_d). We then derive the equilibrium results for Model OW using backward induction, which are detailed in Lemma 4.

Lemma 4. In Model OW, the manufacturer's most advantageous pricing strategies and resultant profits are delineated as follows:

$$w_n = \frac{ep_t+c_n+1}{2},$$

$$p_c = \delta + \frac{v}{2},$$

$$\pi_M^{OW} = \frac{\alpha((-ep_t-c_n+2)\beta+(E+e)p_t+\Delta+c_n-2)(2(ep_t-c_n+1-\delta)+v)}{16(\beta-1)} - \frac{\alpha((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n)((-\beta e+E-e)p_t+(-c_n+2)\beta+\Delta-c_n)}{32(\beta-1)\beta} - p_t\left(e\left(\frac{\alpha((-ep_t-c_n+2)\beta+(E+e)p_t+\Delta+c_n-2)}{8(\beta-1)} - \frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)}{4(2\delta-1)}\right) - Q\right);$$

the remanufacturer's most favorable pricing strategies and corresponding profits are as detailed below:

$$w_r = \frac{(\beta e-E+e)p_t+(c_n+2)\beta+c_n-\Delta}{4},$$

$$\omega = \frac{(\beta e-E+e)p_t+(\beta+1)c_n-\Delta}{2},$$

$$\pi_R^{OW} = \frac{-\alpha((\beta e+E-e)p_t+(\beta-1)c_n+\Delta)^2+16Q\beta p_t(\beta-1)}{16(\beta-1)\beta};$$

the retailer's most favorable pricing strategies and corresponding profits are as detailed below:

$$p_d = \frac{6\delta+v}{4},$$

$$p_n = \frac{ep_t+c_n+3}{4},$$

$$p_r = \frac{(\beta e-E+e)p_t+(c_n+6)\beta-\Delta+c_n}{8},$$

$$\pi_T^{OW} = \frac{\alpha((-ep_t-c_n+2)\beta+(E+e)p_t+\Delta+c_n-2)(-ep_t-c_n+1)}{32(\beta-1)} + \frac{-\alpha(ep_t+2\delta-s-v+c_n-1)^2(\alpha-1)}{16(1-2\delta)} + \frac{\alpha((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n)((-\beta e+E-e)p_t+(-c_n+2)\beta+\Delta-c_n)}{64(\beta-1)\beta};$$

the needs of new and old customers are:

$$d_{nn} = \frac{\alpha((-ep_tc_n+2)\beta+(E+e)p_t+\Delta+c_n-2)}{8(\beta-1)},$$

$$d_{nr} = \frac{\alpha((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n)}{8(1-\beta)\beta},$$

$$d_{on} = \frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)}{4(1-2\delta)}.$$

4.2.2 Consumer subsidy

In the outsourced remanufacturing mode with a CS policy (Model OC), the formulas for calculating the profits of M, R, and T are as outlined below:

$$\pi_M^{OC} = (w_n-c_n)d_{nn}^{OC} + (w_n-c_n-p_c+v)d_{on}^{OC} + (w_r-\omega)d_{nr}^{OC} - p_t(e(d_{nn}^{OC}+d_{on}^{OC})-Q);$$

$$\pi_R^{OC} = (\omega-c_n+\Delta)d_{nr}^{OC} - p_t((e-E)d_{nr}^{OC}-Q);$$

$$\pi_T^{OC} = (p_n-w_n)d_{nn}^{OC} + (p_n-w_n+p_c-p_d)d_{on}^{OC} + (p_r-w_r)d_{nr}^{OC}.$$

The game's sequence follows the same pattern as described in Section 4.2.1. Using backward induction, we derive the equilibrium results for Model OC, which are detailed in Lemma 5.

Lemma 5. In Model OC, the manufacturer's most advantageous pricing strategies and resultant profits are delineated as follows:

$$p_c = \frac{-\beta(v+1)+v+r_a+1}{2\,(1-\beta)},$$

$$w_n = \frac{(\beta-1)ep_t+\beta(c_n+1)-c_n-r_a-1}{2\,(\beta-1)},$$

$$\pi_M^{OC} = \frac{((ep_t-c_n+1)\beta-ep_t+c_n-r_a-1)\alpha((-ep_t-c_n+2)\beta+(E+e)p_t+\Delta+c_n-r_a-2)}{16(\beta-1)^2} + \frac{(\alpha-1)(ep_t+v-c_n)(ep_t-v+c_n)}{8(2\delta-1)} - \frac{\alpha\left((-ep_t-c_n+2)\beta^2+(Ep_t+\Delta-r_a-2)\beta+(e-E)p_t-\Delta+c_n-r_a\right)((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n+r_a)}{32\beta(\beta-1)^2} - p_t\left(e\left(\frac{\alpha((-ep_t-c_n+2)\beta+(E+e)p_t+\Delta+c_n-r_a-2)}{8(\beta-1)} + \frac{(ep_t-v+c_n)(\alpha-1)}{4(2\delta-1)}\right) - Q\right);$$

the remanufacturer's most favorable pricing strategies and corresponding profits are as detailed below:

$$w_r = \frac{(ep_t+c_n+2)\beta^2+(-Ep_t-\Delta+r_a-2)\beta+(E-e)p_t+\Delta-c_n-3r_a}{4(\beta-1)},$$

$$\omega = \frac{(\beta e-E+e)p_t+(\beta+1)c_n+r_a-\Delta}{2},$$

$$\pi_R^{OC} = \frac{-\alpha\left((\beta e+E-e)p_t+\beta c_n+\Delta-c_n+r_a\right)^2+16Q\beta p_t(\beta-1)}{16(\beta-1)\beta};$$

the retailer's most favorable pricing strategies and corresponding profits are as detailed below:

$$p_d = \frac{(v+3)\beta - v - 3r_a - 3}{4(\beta-1)},$$

$$p_n = \frac{(ep_t+c_n+3)\beta - ep_t - c_n - 3r_a - 3}{4(\beta-1)},$$

$$p_r = \frac{(ep_t+c_n+6)\beta^2 + (-Ep_t-\Delta+r_a-6)\beta + (E-e)p_t + \Delta - c_n - 7r_a}{8(\beta-1)},$$

$$\pi_T^{OC} = \frac{\left((-ep_t-c_n+1)\beta+ep_t+c_n-r_a-1\right)\alpha\left((-ep_t-c_n+2)\beta+(E+e)p_t+\Delta+c_n-r_a-2\right)}{32(\beta-1)^2} + \frac{(ep_t-v+c_n)^2(1-\alpha)}{16(2\delta-1)} - \frac{\left((-ep_t-c_n+2)\beta^2+(Ep_t+\Delta-r_a-2)\beta+(e-E)p_t-\Delta+c_n-r_a\right)\alpha\left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n+r_a\right)}{64\beta(\beta-1)^2};$$

the needs of new and old customers are:

$$d_{nn} = \frac{\alpha\left((-ep_t-c_n+2)\beta+(E+e)p_t+\Delta+c_n-r_a-2\right)}{8(\beta-1)},$$

$$d_{nr} = \frac{\alpha\left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n+r_a\right)}{8(1-\beta)\beta},$$

$$d_{on} = \frac{(1-\alpha)(ep_t-v+c_n)}{4(1-2\delta)}.$$

4.2.3 Remanufacturer subsidy

In the outsourced remanufacturing mode with a RS policy (Model OR), the formulas for calculating the profits of M, R, and T are as outlined below:

$$\pi_M^{OR} = (w_n - c_n)d_{nn}^{OR} + (w_n - c_n - p_c + v)d_{on}^{OR} + (w_r - \omega)d_{nr}^{OR} - p_t(e(d_{nn}^{OR} + d_{on}^{OR}) - Q);$$

$$\pi_R^{OR} = (\omega + r_b - c_n + \Delta)d_{nr}^{OR} - p_t((e-E)d_{nr}^{OR} - Q);$$

$$\pi_T^{OR} = (p_n - w_n)d_{nn}^{OR} + (p_n - w_n + p_c - p_d)d_{on}^{OR} + (p_r - w_r)d_{nr}^{OR}.$$

The game's sequence follows the same pattern as described in Section 4.2.1. Using backward induction, we derive the equilibrium results for Model OR, which are detailed in Lemma 6.

Lemma 6. In Model OR, the manufacturer's most advantageous pricing strategies and resultant profits are delineated as follows:

$$p_c = \delta + \frac{v}{2},$$

$$w_n = \frac{ep_t+c_n+1}{2},$$

$$\pi_M^{OR} = \frac{(ep_t-c_n+1)\left((-\beta e+E+e)p_t+(-c_n+2)\beta+\Delta+c_n+r_b-2\right)\frac{\alpha}{16(\beta-1)}+(-p_t+2\delta-v+c_n-1)(ep_t+2\delta-v+c_n-1)(\alpha-1)}{8(\delta-1)} - \frac{\alpha\left((-ep_t-c_n+2)\beta+(E-e)p_t+\Delta-c_n+r_b\right)\left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n+r_b\right)}{32(\beta-1)\beta} - p_t\left(e\left(\frac{\alpha\left((-ep_t-c_n+2)\beta+(E+e)p_t+\Delta+c_n+r_b-2\right)}{8(\beta-1)} - \frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)}{4(2\delta-1)}\right) - Q\right);$$

the remanufacturer's most favorable pricing strategies and corresponding profits are as detailed below:

$$w_r = \frac{(\beta e - E + e)p_t + (c_n+2)\beta + c_n - r_b - \Delta}{4},$$

$$\omega = \frac{(\beta e - E + e)p_t + (\beta+1)c_n - r_b - \Delta}{2},$$

$$\pi_R^{OR} = \frac{-\alpha\left((\beta e + E - e)p_t + \beta c_n + \Delta - c_n + r_b\right)^2 + 16Q\beta p_t(\beta-1)}{16\beta(\beta-1)};$$

the retailer's most favorable pricing strategies and corresponding profits are as detailed below:

$$p_d = \frac{3\delta}{2} + \frac{v}{4},$$

$$p_n = \frac{ep_t + c_n + 3}{4},$$

$$p_r = \frac{(\beta e - E + e)p_t + (c_n + 6)\beta + c_n - r_b - \Delta}{8},$$

$$\pi_T^{OR} = -\frac{(ep_t + c_n - 1)\big((-\beta e + E + e)p_t + (-c_n + 2)\beta + \Delta + c_n + r_b - 2\big)\alpha}{32(\beta - 1)} + \frac{(ep_t + 2\delta - v + c_n - 1)^2(\alpha - 1)}{16(2\delta - 1)} - \frac{\alpha\big((-ep_t - c_n + 2)\beta + (E - e)p_t + \Delta - c_n + r_b\big)\big((ep_t + c_n)\beta + (E - e)p_t + \Delta - c_n + r_b\big)}{64(\beta - 1)\beta};$$

the needs of new and old customers are:

$$d_{nn} = \frac{(-ep_t - c_n + 2)\beta + (E + e)p_t + \Delta + c_n + r_b - 2}{8(\beta - 1)},$$

$$d_{nr} = \frac{\alpha\big((ep_t + c_n)\beta + (E - e)p_t + \Delta - c_n + r_b\big)}{8(1 - \beta)\beta},$$

$$d_{on} = \frac{(\alpha - 1)(ep_t + 2\delta - v + c_n - 1)}{4(1 - 2\delta)}.$$

5. Model comparison and analysis

In this section, we compare the equilibrium results of each game to study the impact of carbon emission control and subsidy policies on the optimal remanufacturing mode choice and pricing decisions of OEM manufacturers, as well as the selection of the optimal subsidy method.

5.1 Comparative analysis of subsidy policies under the authorized remanufacturing model

Theorem 1. In models AW, AC, and AR, the optimal prices for manufacturers, remanufacturers, and retailers satisfy the following relationships:

(i) When $r_a > r_a^A$, $w_n^{AW} < w_n^{AC}$, $w_r^{AW} < w_r^{AC}$, $p_n^{AW} < p_n^{AC}$, $p_r^{AW} < p_r^{AC}$, $p_c^{AW} < p_c^{AC}$, $p_d^{AW} < p_d^{AC}$;

(ii) $p_n^{AW} < p_n^{AR}$, $w_n^{AW} < w_n^{AR}$, $p_c^{AW} < p_c^{AR}$, $p_d^{AW} < p_d^{AR}$, $w_r^{AW} > w_r^{AR}$, $p_r^{AW} > p_r^{AR}$;

(iii) When $e < e_1^A$, $w_n^{AC} > w_n^{AR}$, $w_r^{AC} > w_r^{AR}$, $p_n^{AC} > p_n^{AR}$, $p_r^{AC} > p_r^{AR}$, $p_c^{AC} > p_c^{AR}$, $p_d^{AC} > p_d^{AR}$. Where r_a^A (e_1^A) represents a threshold related to r_a (e), which is detailed in the Appendix and consistently addressed throughout the paper.

Theorem 1(i) indicates that, compared to the no-subsidy policy, under the CS policy, a firm's pricing decisions are influenced by the government's subsidy amount to consumers. When the CS exceeds a certain threshold, OEMs and authorized remanufacturers (CMs) will increase the wholesale and retail prices of both new and remanufactured products to offset the effect of the government subsidies. Simultaneously, OEMs will raise the discount prices for trade-in schemes to encourage consumers to purchase new products through trade-ins. Theorem 1(ii) shows that under remanufacturer subsidies, OEMs will set the wholesale and retail prices for new products and trade-in discounts higher than under no-subsidy policies. In contrast, CM's wholesale and retail prices for remanufactured products will be lower under remanufacturer subsidies compared to no-subsidy policies. This occurs because when the government grants subsidies to CMs, they are more likely to enhance product quality to increase sales and improve market acceptance and profitability of remanufactured products. Theorem 1(iii) indicates that the firm's pricing decisions under CS policies versus RS policies primarily depend on the carbon emissions per unit of new products. When these emissions are below a certain threshold, the wholesale and retail prices of new products and the trade-in discounts under CS policies are always higher than under RS policies. This suggests that high-carbon-emission new products are less favored in markets with increasing environmental awareness, and consumers prefer low-carbon remanufactured products. Additionally, trade-in incentives drive more old products into remanufacturing. Remanufacturer subsidies mainly indirectly offset the cost of producing each unit of remanufactured product, having a relatively weaker effect on market demand compared to consumer subsidies. Therefore, we can draw some management implications from the above analysis:

(1) The government's CS strategy is most effective in authorized remanufacturing models. When designing CS policies, the government should consider the threshold of subsidy amounts to avoid excessive price increases that could negate the subsidy effects. Appropriate subsidy levels should be set to encourage consumers to choose low-carbon products.

(2) Authorized remanufacturers should focus on improving the quality and performance of remanufactured products to expand market share. High-quality remanufactured products will enhance consumer acceptance and loyalty, leading to increased sales and profits.

(3) Invest in marketing strategies to improve consumer awareness of remanufactured products, emphasizing their quality, reliability, and environmental benefits. Building consumer trust is crucial for increasing acceptance and demand.

Theorem 2. In models AW, AC, and AR, the profit relations for manufacturers, remanufacturers, and retailers are as follows:

(i) When $r_{b_1}^A < r_b < r_{b_2}^A$, $\pi_M^{AR} > \pi_M^{AW}$; When $r_{a_1}^A < r_a < r_{a_2}^A$, $\pi_M^{AC} > \pi_M^{AR}$, $\pi_M^{AC} > \pi_M^{AW}$.

(ii) When $r_{b_3}^A < r_b < r_{b_4}^A$, $\pi_R^{AR} > \pi_R^{AW}$; When $r_{a_3}^A < r_a < r_{a_4}^A$, $\pi_R^{AC} > \pi_R^{AR}$, $\pi_T^{AC} > \pi_T^{AW}$.

(iii) When $r_{b_5}^A < r_b < r_{b_6}^A$, $\pi_T^{AR} > \pi_T^{AW}$; When $r_{a_5}^A < r_a < r_{a_6}^A$, $\pi_T^{AC} > \pi_T^{AR}$, $\pi_T^{AC} > \pi_T^{AW}$.

Theorem 2 indicates that, in authorized remanufacturing models, compared to no-subsidy policies, both excessively high and low remanufacturer or consumer subsidy amounts will result in lower profits for supply chain members compared to no-subsidy scenarios. Only when the government subsidies for remanufacturers (or consumers) are within a specific threshold range will the profits of supply chain members under these subsidy policies exceed those under no-subsidy policies. Furthermore, consumer subsidies are only optimal if the subsidy amount reaches a certain threshold. Although the precise thresholds for these relationships are complex and yield few direct managerial insights, extensive numerical simulations have been conducted to provide more actionable guidance for managers and policymakers. We deliberately select parameters to encompass a broad spectrum of relevant outcomes. Considering that the focus is not on market size, production costs for new products, cost savings per unit of remanufactured product or the carbon emissions generated by a new product, we fix these parameters at $c_n = 0.2$, $\Delta = 0.1$, $v = 0.5$, $\alpha = 0.5$, $e = 2$, and $Q = 0.6$. We focus on the other five parameters that are interesting to analyze, namely, the durability of the product δ, consumer acceptance of remanufactured products β, carbon emissions reductions per remanufactured unit E, CET prices p_t, consumer subsidies r_a, and remanufacturer subsidies r_b. According to Chai et al. (2023), We examined three distinct values for each of the parameters. Given that $0 < \beta < 1$, we set $\beta \in \{0.2, 0.6, 0.9\}$; δ must remain below β (i.e., $0 < \delta < \beta < 1$), otherwise, the CM would opt out of remanufacturing; hence, we select δ values of $\{0.01, 0.05, 0.08\}$. The reduction in carbon emissions should not surpass the actual carbon emissions, leading us to consider E values of $\{0.02, 0.06, 0.1\}$. While there is no strict upper limit on the CET price, it should not be excessively high, so we choose p_t values of $\{0.1, 0.5, 1\}$. Lastly, government subsidies should not exceed production costs; therefore, we set r_a and r_b values at $\{0.02, 0.06, 0.09\}$. This approach results in a total of 729 parameter combinations. We computed the variations in consumer surplus for each scenario, and the findings are summarized in Tables 2, 3, and 4.

Table 2
Frequency of manufacturers' profit differentials

	Condition	Times	Frequency (%)
$\pi_M^{AW} - \pi_M^{AR}$	>0	0	0
	=0	0	0
	<0	729	100
$\pi_M^{AW} - \pi_M^{AC}$	>0	45	6.17
	=0	0	0
	<0	684	93.83
$\pi_M^{AR} - \pi_M^{AC}$	>0	0	0
	=0	0	0
	<0	729	100

Table 3
Frequency of profit difference of the remanufacturer

	Condition	Times	Frequency (%)
$\pi_R^{AW} - \pi_R^{AR}$	>0	141	19.34
	=0	0	0
	<0	588	80.66
$\pi_R^{AW} - \pi_R^{AC}$	>0	135	18.52
	=0	0	0
	<0	594	81.48
$\pi_R^{AR} - \pi_R^{AC}$	>0	225	30.86
	=0	0	0
	<0	504	69.14

Table 4
Frequency of margin differences among retailers

	Condition	Times	Frequency (%)
$\pi_T^{AW} - \pi_T^{AR}$	>0	225	30.86
	=0	0	0
	<0	504	69.14
	>0	0	0

$\pi_T^{AW} - \pi_T^{AC}$	=0	0	0
	<0	729	100
	>0	0	0
$\pi_T^{AR} - \pi_T^{AC}$	=0	0	0
	<0	729	100

The CS policies are beneficial for both manufacturers and retailers and are often advantageous for remanufacturers as well. Manufacturers and retailers under CS policies have profits higher than or equal to those under no-subsidy (or RS) policies in 729 cases, representing 100% of all scenarios. Remanufacturers earn higher profits under consumer subsidies than no-subsidy policies in 594 cases and higher than under RS policies in 504 cases, representing 81.48% and 69.14% respectively. Supply chain members generally prefer CS policies, as they yield higher profits in almost all cases, making consumer subsidies an effective policy from a profit perspective.

Drawing from the work of Zhu et al. (2017) and according to Assumption 4, which specifies the utility functions for consumers purchasing new and remanufactured products, the consumer surplus under no subsidy policy and RS policy is $\alpha \int_{\frac{p_n-p_r}{1-\beta}}^{1}(\theta - p_n)d\theta + \alpha \int_{\frac{p_r}{\beta}}^{\frac{p_n-p_r}{1-\beta}} (\beta\theta - p_r)d\theta + (1-\alpha) \int_{\frac{p_n-p_d}{1-2\delta}}^{1}((1-\delta)\theta - p_n + p_d)d\theta$, and under CS policy, the consumer surplus is $\alpha \int_{\frac{p_n-p_r}{1-\beta}}^{1}(\theta - p_n)d\theta + \alpha \int_{\frac{p_r-r_a}{\beta}}^{\frac{p_n+r_a-p_r}{1-\beta}} (\beta\theta + r_a - p_r)d\theta + (1-\alpha) \int_{\frac{p_n-p_d}{1-2\delta}}^{1}((1-\delta)\theta - p_n + p_d)d\theta$.

Theorem 3. In models AW, AC, and AR, consumer surplus is determined as follows: When $r_b > \frac{(ep_t+c_n)\beta^2+((E+e)p_t+\Delta+c_n-6)\beta-(4E+2e)p_t}{6} - \frac{2\Delta+c_n}{3} + 1$, $CS^{AW} < CS^{AC} < CS^{AR}$; otherwise, $CS^{AR} < CS^{AW} < CS^{AC}$.

Theorem 3 compares consumer surpluses under three government subsidy policies in authorized remanufacturing models. It is found that consumer surplus in model AC (consumer subsidies) is not always greater than in models AW (no subsidy) or AR (remanufacturer subsidies). Consumer surplus varies with the level of government subsidies to products; specifically, if subsidies to remanufacturers are high, they benefit consumers. Conversely, if subsidies to consumers are high, consumer surplus is maximized under the CS policy.

Based on the findings by Xia et al. (2020), the environmental effects when applying a government subsidy strategy can be assessed as follows: $EI = e(d_{nn}^* + d_{on}^*) + (e - E)d_{nr}^*$.

Theorem 4. In models AW, AC, and AR, environmental benefits are given by: When $e \geq \frac{(-\beta^2 c_n + (-Ep_t - \Delta - c_n - r_b)\beta - 2Ep_t + v - 2\Delta + c_n - 2r_b)\alpha - v + c_n}{(-1+(\beta^2+\beta-1)\alpha)p_t}$, $EI^{AW} < EI^{AC} < EI^{AR}$; when $\frac{E(\beta+2)}{2(1-\beta)} < e < \frac{(-\beta^2 c_n + (-Ep_t - \Delta - c_n - r_b)\beta - 2Ep_t + v - 2\Delta + c_n - 2r_b)\alpha - v + c_n}{(-1+(\beta^2+\beta-1)\alpha)p_t}$, $EI^{AR} < EI^{AW} < EI^{AC}$.

Theorem 4 suggests that in authorized remanufacturing models, environmental costs are influenced by the carbon emissions per unit of new products rather than the choice of subsidy recipient. Specifically, when carbon emissions per new product are high, policymakers should consider subsidizing remanufacturers to achieve better environmental performance. Conversely, when emissions are low, consumer subsidies are preferred to improve environmental outcomes.

5.2. Comparative analysis of subsidy policies under the outsourcing remanufacturing model

Theorem 5. In models OW, OC, and OR, the optimal prices for manufacturers, remanufacturers, and retailers satisfy the following:

(i) $w_n^{OW} < w_n^{OC}$, $w_r^{OW} < w_r^{OC}$, $p_n^{OW} < p_n^{OC}$, $p_r^{OW} < p_r^{OC}$, $p_c^{OW} < p_c^{OC}$, $p_d^{OW} < p_d^{OC}$;

(ii) $p_n^{OW} = p_n^{OR}$, $w_n^{OW} = w_n^{OR}$, $p_c^{OW} = p_c^{OR}$, $p_d^{OW} = p_d^{OR}$, $w_r^{OW} > w_r^{OR}$, $p_r^{OW} > p_r^{OR}$;

(iii) $w_n^{OC} > w_n^{OR}$, $w_r^{OC} > w_r^{OR}$, $p_n^{OC} > p_n^{OR}$, $p_r^{OC} > p_r^{OR}$, $p_c^{OC} > p_c^{OR}$, $p_d^{OC} > p_d^{OR}$.

Theorem 5 compares the effects of no-subsidy (OW), consumer subsidy (OC), and remanufacturing subsidy (OR) policies. The analysis shows that under CS policies, the wholesale and retail prices for both new and remanufactured products, as well as trade-in discounts, are the highest. Government subsidies to remanufacturers do not affect wholesale and retail prices or trade-in discounts for new products. However, consumer subsidies typically lead to higher wholesale prices for remanufactured products as CMs offset the impact of the subsidies. Conversely, subsidies to remanufacturers often result in improved product quality and increased market acceptance. Therefore, we can draw some management implications from the above analysis:

(1) In outsourced remanufacturing models, consumer subsidies effectively boost demand for remanufactured products and are therefore an effective strategy to promote the remanufacturing industry.

(2) Under CS policies, OEMs should adjust wholesale and retail prices of new and remanufactured products to maintain competitiveness and enhance trade-in discounts to expand market share.

(3) Remanufacturers should focus on improving product quality and innovation to retain market share under remanufacturer subsidies.

(4) Governments should support trade-in programs by providing financial incentives to consumers, thereby increasing the supply of old products for remanufacturing, promoting the circular economy, and reducing waste.

Theorem 6. In models OW, OC, and OR, the profit relationships for manufacturers, remanufacturers, and retailers are:

(i) When $r_{b_1}^O < r_b < 1$, $\pi_M^{OR} > \pi_M^{OW}$; when $r_{a_1}^O < r_a < r_{a_2}^O$, $\pi_M^{OR} > \pi_M^{OC}$; when $r_a > r_{a_3}^O$, $\pi_M^{OC} > \pi_M^{OW}$.

(ii) When $r_{b_2}^O < r_b < 1$, $\pi_R^{OR} > \pi_R^{OW}$; when $r_b < r_a < r_{a_4}^O$, $\pi_R^{OC} > \pi_R^{OR}$; when $r_a > r_{a_5}^O$, $\pi_R^{OC} > \pi_R^{OW}$.

(iii) When $r_{b_3}^O < r_b < 1$, $\pi_T^{OR} > \pi_T^{OW}$; when $r_{a_6}^O < r_a < r_{a_7}^O$, $\pi_T^{OC} > \pi_T^{OR}$; when $r_a > r_{a_8}^O$, $\pi_T^{OC} > \pi_T^{OW}$.

Theorem 6 demonstrates that in the outsourcing remanufacturing model when the level of RS (or CS) surpasses a specific threshold, the profits of supply chain members under the RS (or CS) policy are higher than those under the anarchic subsidy policy. By comparing the profit relationship between CS and RS policy, it is found that when the amount of CS is in a certain threshold range, the profit of supply chain members under CS policy is higher than that under RS policy. While we have obtained a decisive threshold for their relationship, the expression of closed forms is complex and leads to few management insights. To help managers and government decision-makers gain more useful management insights, we conducted extensive numerical simulations to try to find out which policies are more likely to benefit supply chain members in an outsourced remanufacturing model. The parameters we selected were the same as in the previous simulation, and we calculated the difference in corporate profits under each combination. A summary of the results is presented in Tables 5, 6, and 7.

Table 5
Frequency of manufacturers' profit differentials

	Condition	Times	Frequency (%)
$\pi_M^{OW} - \pi_M^{OR}$	>0	105	14.40
	=0	0	0
	<0	624	85.60
$\pi_M^{OW} - \pi_M^{OC}$	>0	231	31.69
	=0	0	0
	<0	498	68.31
$\pi_M^{OR} - \pi_M^{OC}$	>0	327	44.86
	=0	0	0
	<0	402	55.14

Table 6
Frequency of profit difference of remanufacturer

	Condition	Times	Frequency (%)
$\pi_R^{OW} - \pi_R^{OR}$	>0	171	23.46
	=0	9	1.23
	<0	549	75.31
$\pi_R^{OW} - \pi_R^{OC}$	>0	171	23.46
	=0	9	1.23
	<0	549	75.31
$\pi_R^{OR} - \pi_R^{OC}$	>0	243	33.33
	=0	243	33.33
	<0	243	33.33

Table 7
Frequency of margin differences among retailers

	Condition	Times	Frequency (%)
$\pi_T^{OW} - \pi_T^{OR}$	>0	231	31.69
	=0	0	0
	<0	498	68.31
$\pi_T^{OW} - \pi_T^{OC}$	>0	243	33.33
	=0	0	0

	<0	486	66.67
$\pi_T^{OR} - \pi_T^{OC}$	>0	0	0
	=0	0	0
	<0	729	100

Under the outsourcing remanufacturing model, in most cases, it is beneficial for both manufacturers and retailers. There were 498 cases (402 cases) in which the profit of manufacturers under the CS policy was higher or equal to that without the subsidy policy (or RS policy), accounting for 68.31% (55.14%) of all cases. There were 486 cases (729 cases) in which the retailer's profit under the CS policy was higher or equal to that without the subsidy policy (or RS policy), accounting for 66.67% (100%) of all cases. Remanufacturers prefer subsidy policies (both CS policies and RS policies) because in almost all cases, profits under CS policies or RS policies are higher than without policy.

Theorem 7. In models OW, OC, and OR, consumer surplus is determined as follows: When $r_b > (-2\beta e - 2E + 2e)p_t + (-2\beta + 2)c_n - 2\Delta$, $CS^{OW} < CS^{OC} < CS^{OR}$; Otherwise, $CS^{OR} < CS^{OW} < CS^{OC}$.

Theorem 7 indicates that consumer surplus varies with the level of government subsidies. Specifically, consumer surplus under remanufacturer subsidies is maximized only if the subsidy amount exceeds a certain threshold; otherwise, consumer subsidies yield the highest consumer surplus. This approach needs to be carried out within the limits set by CET policies, enabling consumers to acquire remanufactured products at discounted prices, which directly boosts the demand for remanufactured goods and is more favorable for achieving consumer surplus compared to remanufacturer subsidies.

Theorem 8. In models OW, OC, and OR, environmental benefits satisfy: When $e \geq \frac{E(r_a - r_b)}{(\beta+1)r_a + r_b(\beta-1)}$, $EI^{OW} < EI^{OC} < EI^{OR}$; when $\frac{E}{1-\beta} \leq e < \frac{E(r_a - r_b)}{(\beta+1)r_a + r_b(\beta-1)}$, $EI^{OR} < EI^{OW} < EI^{OC}$.

Theorem 8 indicates that the impact of government policies on environmental benefits is influenced by carbon emissions levels. As consumer willingness to purchase new products increases, the sales volume of new products rises, leading to higher overall carbon emissions in the manufacturing industry. Conversely, as consumer willingness to purchase remanufactured products grows, the sales volume of remanufactured products increases, resulting in lower overall carbon emissions in the manufacturing industry. Therefore, compared to subsidies for remanufacturers, consumer subsidies can generate greater environmental benefits.

5.3 Comparative analysis of different remanufacturing modes under carbon emission control and subsidy policies

Theorem 9. In models AW and OW, the optimal prices for manufacturers, remanufacturers, and retailers satisfy the following relationships:

(i) When $e < \frac{Ep_t + \beta c_n + \Delta - c_n}{p_t(1-\beta)}$, $w_n^{AW} > w_n^{OW}$, $w_r^{AW} > w_r^{OW}$, $p_n^{AW} > p_n^{OW}$, $p_r^{AW} > p_r^{OW}$, $p_c^{AW} > p_c^{OW}$, $p_d^{AW} > p_d^{OW}$;

(ii) When $r_a > \frac{4(\beta-1)(\frac{1}{4}(-5ep_t - 5c_n + 9)\beta + (E-e)p_t + \Delta - c_n)}{5\beta+4}$, $w_n^{AC} > w_n^{OC}$, $w_r^{AC} > w_r^{OC}$, $p_n^{AC} > p_n^{OC}$, $p_r^{AC} > p_r^{OC}$, $p_c^{AC} > p_c^{OC}$, $p_d^{AC} > p_d^{OC}$;

(iii) When $r_b > (-\beta e - E + e)p_t + (1-\beta)c_n - \Delta$, $w_n^{AR} > w_n^{OR}$, $w_r^{AR} > w_r^{OR}$, $p_n^{AR} > p_n^{OR}$, $p_r^{AR} > p_r^{OR}$, $p_c^{AR} > p_c^{OR}$, $p_d^{AR} > p_d^{OR}$.

Theorem 9 compares the price relationships under different government subsidy policies for two remanufacturing modes. Theorem 9(i) shows that without subsidy policies, a company's pricing decisions are influenced by the carbon emissions per unit of new products. Specifically, in the absence of subsidies, companies bear the carbon emission costs associated with production. When the carbon emissions per unit of new products are lower, the carbon costs are lower. Therefore, in an authorized remanufacturing mode, companies can be more flexible in pricing and have greater pricing power. In contrast, the outsourced remanufacturing mode typically involves agreements with third parties, where pricing must consider the profit margins of the outsourcing party and market competition, resulting in less pricing flexibility.

Theorem 9(ii) and (iii) show that under consumer subsidies and remanufacturer subsidies, a company's pricing decisions are influenced by the subsidy amounts. Specifically, under consumer subsidies, since new and remanufactured products are distributed through different channels in the authorized mode, original manufacturers can leverage their brand influence and market promotion to attract consumers to buy new products, thus the price of new products is relatively higher. Remanufacturers, on the other hand, must pay authorization fees and have higher production costs, so they pass these costs onto the price of remanufactured products, making them relatively higher. When the consumer subsidy amounts are higher, consumer subsidies offset part of the remanufactured product prices, thus increasing consumer willingness to purchase remanufactured products. That is, even if the price increases, consumers are willing to buy. Under RS policies, companies directly manage and control the remanufacturing process in the authorized mode, including production techniques, quality, and costs, allowing them to convert subsidies into higher prices and greater profit margins. Authorized remanufacturing enables companies to optimize production processes, enhance the added value of remanufactured products, and maintain higher flexibility and brand premium in pricing strategies. In contrast, under the outsourced model, the impact of subsidies is shared with third-party remanufacturers, resulting in weaker control over pricing and costs.

From the above analysis, some managerial insights can be drawn:

(1) When carbon emission costs are lower, the authorized remanufacturing mode provides a pricing advantage for companies. However, when carbon emission costs are higher or market uncertainty is greater, the outsourced remanufacturing mode is a more economical choice.

(2) Companies should focus on improving low-carbon production technologies to reduce carbon emissions per unit of product during production. This not only reduces carbon costs and enhances pricing capability under no-subsidy policies but also strengthens the company's market image and competitiveness in high-end markets.

(3) Under CS policies or RS policies, prioritizing the authorized remanufacturing mode maximizes subsidy benefits, especially when subsidy amounts are higher.

Theorem 10. In models AW and OW, the profits of manufacturers, remanufacturers, and retailers satisfy the following relationships:

(i) When $e > \max\{e_1^W, e_2^W, e_3^W\}$, $\pi_M^{AW} > \pi_M^{OW}$, $\pi_R^{AW} > \pi_R^{OW}$, $\pi_T^{AW} > \pi_T^{OW}$.

(ii) When $\max\{r_{a_1}, r_{a_2}, r_{a_3}\} < r_a < \min\{r_{a_4}, r_{a_5}, r_{a_6}\}$, $\pi_M^{AC} > \pi_M^{OC}$, $\pi_R^{AC} > \pi_R^{OC}$, $\pi_T^{AC} > \pi_T^{OC}$.

(iii) $\pi_M^{AR} > \pi_M^{OR}$; $\pi_R^{AR} > \pi_R^{OR}$; When $r_{b_1} < r_b < r_{b_2}$, $\pi_T^{AR} > \pi_T^{OR}$.

Theorem 10 compares the profit relationships under different government subsidy policies for two remanufacturing modes. Theorem 10(i) shows that without subsidy policies, when the carbon emissions per unit of new products exceed a certain threshold, manufacturers, remanufacturers, and retailers can achieve mutual benefits in the authorized remanufacturing mode. This is because in the authorized remanufacturing mode, manufacturers have direct control over the remanufacturing process, enabling them to use production technologies and resources more effectively to reduce overall carbon emissions. Additionally, authorized remanufacturing allows manufacturers to better optimize product design, making the remanufacturing process more efficient and environmentally friendly, thus further reducing production and environmental costs. In contrast, in the outsourced remanufacturing mode, remanufacturers often operate independently, and manufacturers cannot directly control the quality and efficiency of the remanufacturing process, leading to additional coordination costs and unnecessary waste. Therefore, under high carbon emission conditions, authorized remanufacturing can better reduce total costs and achieve mutual benefits.

Theorem 10(ii) shows that under CS policies, when the consumer subsidy amount is within a certain threshold range, manufacturers, remanufacturers, and retailers can achieve mutual benefits in the authorized remanufacturing mode. In this scenario, consumer subsidies lower the final market price of remanufactured products, boosting product market demand. In the authorized remanufacturing mode, manufacturers can respond more quickly to market demand changes, ensure product quality, and lower unit costs through economies of scale. Moreover, manufacturers in the authorized mode can better integrate marketing channels and brand image, enhancing market competitiveness. In contrast, in the outsourced remanufacturing mode, remanufacturers have greater autonomy, but manufacturers' response speed to market changes is slower, and outsourcing costs are harder to control, which can erode some profits.

Theorem 10(iii) shows that under RS policies, profits are higher for manufacturers and remanufacturers in the authorized remanufacturing mode, and when the subsidy amount is within a certain threshold range, profits for retailers are higher. This is because subsidies directly reduce the costs for remanufacturers, allowing them to offer more competitive remanufactured products. In the authorized remanufacturing mode, manufacturers and remanufacturers can share the benefits of cost reductions. Furthermore, manufacturers can control the remanufacturing process through authorization, ensuring product quality and brand consistency while optimizing supply chain management, reducing unnecessary intermediate links and inventory costs. These factors collectively enhance the profits of manufacturers and remanufacturers. When subsidies are within an appropriate threshold range, remanufacturers' costs decrease to a level where retailers can purchase remanufactured products at lower prices. The product quality and supply chain stability in the authorized mode provide better assurance, reducing retailer risk and helping to increase sales and profit margins.

6. Numerical analysis

In this part, MATLAB R2022a is utilized for numerical analysis to examine how CET prices, subsidies provided to consumers for purchasing remanufactured products, and subsidies given to remanufacturers affect environmental outcomes, consumer surplus, social welfare, and the overall profit margins of supply chain entities. Taking IBM's computer hardware and servers as an example, unit-remanufactured computer hardware and servers can save 50% of costs and 65% of resources compared to new ones, reducing environmental impact by 75%. Following Xia et al. (2020), the relevant parameters for remanufactured products are set as $c_n = 0.2$, $\Delta = 0.1$, $v = 0.5$, $\alpha = 0.5$, $\beta = 0.7$, $\delta = 0.6$, $e = 2$, $E = 1$, $s = 1$. Drawing from the work of Zhu et al. (2017), we define social welfare $SW = \pi_M^* + \pi_R^* + \pi_T^* + CS - EI$.

6.1 The influence of p_t and r_a under the consumer subsidy policy

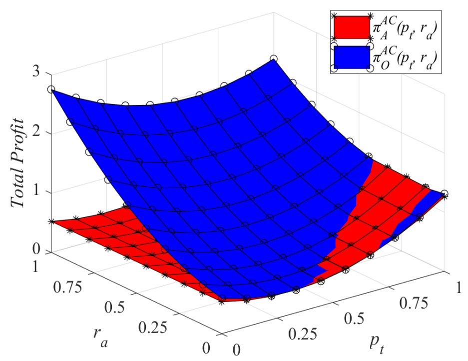


Fig. 3. The impact of p_t and r_a on the total profit of the supply chain under the authorized and outsourcing remanufacturing models

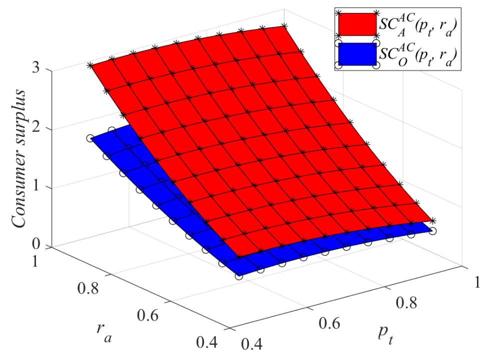


Fig. 4. The impact of $\mathrm{p_t}$ and $\mathrm{r_a}$ on consumer surplus under authorized and outsourcing remanufacturing models

Fig. 3 illustrates that under both authorized and outsourced remanufacturing modes, as CET prices and the consumer subsidy amounts increase, the total profit of the supply chain shows an upward trend. This indicates that under high CET prices and consumer subsidies, both authorized and outsourced remanufacturing modes positively influence supply chain profits. The primary reason is that the increase in CET prices implies higher costs for carbon emissions during production. Consequently, higher CET prices incentivize enterprises to adopt more environmentally friendly production methods or increase the rate of product remanufacturing. Since remanufactured products have lower carbon emissions during the production process, the increase in CET prices significantly boosts the profit of the remanufacturing mode. Consumer subsidies are economic support provided by the government to consumers purchasing remanufactured products, aimed at encouraging consumers to choose remanufactured products. This directly reduces consumer purchasing costs, increases the attractiveness of remanufactured products to consumers, promotes market acceptance, and enhances consumer surplus. Fig. 4 shows that under both authorized and outsourced remanufacturing modes, consumer surplus increases with the increase in CET prices initially but starts to decrease after reaching a certain threshold. This is because as CET prices increase, enterprises need to pay higher carbon emission fees during the remanufacturing process, directly increasing remanufacturing costs. To maintain profits, enterprises may pass on the increased costs to consumers, leading to price hikes and subsequent reductions in consumer surplus. Consumer surplus increases with the increase in consumer subsidies. This is because higher consumer subsidies mean that consumers can receive more economic subsidies, which partly offset the price increases due to increased remanufacturing costs, making remanufactured products more attractive to consumers, promoting market acceptance, and increasing consumer surplus.

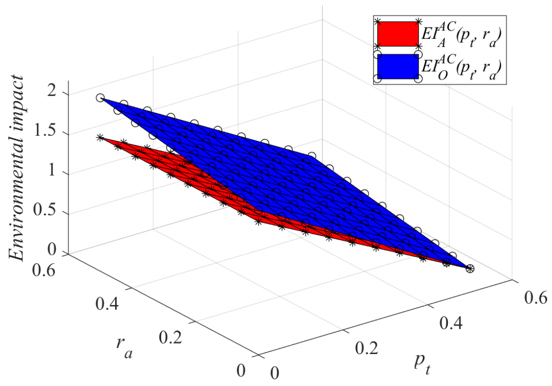


Fig. 5. Environmental impacts of $\mathrm{p_t}$ and $\mathrm{r_a}$ under the authorized and outsourcing remanufacturing models

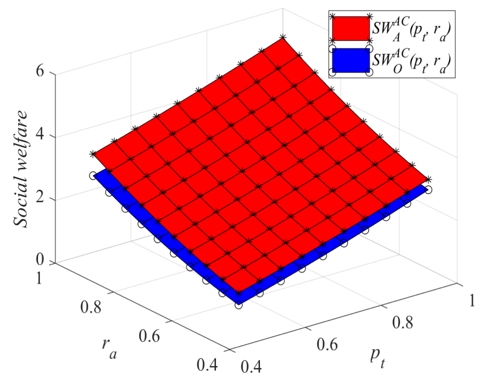


Fig. 6. The impact of $\mathrm{p_t}$ and $\mathrm{r_a}$ on social welfare under the authorized and outsourcing remanufacturing models

Fig. 5 shows that as CET prices increase, environmental impact under authorized remanufacturing mode exhibits a significant downward trend. This indicates that the increase in CET prices can effectively reduce carbon emissions during the remanufacturing process, thus reducing environmental impact. With the increase in government subsidies to consumers, the environmental impact shows a slow downward trend, although not significant, indicating the positive effect of subsidies on the environment. Under the outsourced remanufacturing mode, environmental impact similarly decreases significantly with the increase in CET prices. This consistency with the results of the authorized remanufacturing mode suggests the universality of the reduction in environmental impact with the increase in CET prices. For consumer subsidies, the change in environmental impact under the outsourced mode shows a slower trend, with small changes. This may be because the transmission effect of subsidies under the outsourced mode is weak, failing to significantly affect environmental impact. These phenomena are mainly due to the direct increase in carbon emission costs with the increase in CET prices, prompting enterprises to adopt cleaner production technologies and optimize production processes to reduce carbon emissions, leading to a significant reduction in environmental impact. Government subsidies to consumers mainly affect environmental impact indirectly by stimulating consumption. With the increase in government subsidies to consumers, consumer willingness to purchase remanufactured products enhances, increasing market demand, prompting enterprises to increase remanufacturing production, and reducing environmental impact. Fig. 6 shows that under the authorized remanufacturing mode, as CET prices and remanufacturer subsidies increase, social welfare significantly improves. The increase in CET prices means that enterprises face higher carbon emission costs during production. To reduce costs, enterprises tend to invest more in remanufacturing, thereby reducing carbon emissions. This not only improves environmental quality but also reduces waste pollution and improves resource utilization efficiency, contributing to sustainable development goals and enhancing social welfare. The increase in remanufacturer subsidies allows enterprises to allocate more funds to remanufacturing production, expanding production scale, and increasing the supply of remanufactured products. Through CET and RS policies, governments can guide the economy towards low-carbon and environmentally friendly directions, promoting the development of the green economy. This transformation not only creates new economic growth points but also promotes employment and increases overall social welfare.

6.2 The influence of p_t and r_b under the remanufacturer subsidy policy

Fig. 7 illustrates that in the authorized remanufacturing mode, as CET prices climb, the profit curve exhibits an upward trend but tends to flatten or even decrease at higher CET prices. The increase in government subsidies to remanufacturers has a positive effect on profits. Under the outsourced remanufacturing mode, profits are lower at lower CET prices, gradually increase with the increase in CET prices, and reach a peak before stabilizing. Similar to the authorized remanufacturing mode, an increase in government subsidies to remanufacturers also has a positive effect on profits, but the marginal benefits of subsidies diminish at higher CET prices. The reason lies in the direct impact of CET prices on enterprise production costs and profits. At lower CET prices, carbon costs are lower, allowing enterprises to produce at lower costs, resulting in higher profits. However, as CET prices increase, carbon costs increase, compressing profits. Government subsidies can directly cut costs for remanufacturers, thereby boosting remanufacturing profits. This is essential for advancing the growth of the remanufacturing industry. At lower CET prices, subsidies significantly increase profits, and promote remanufacturing activities, but at higher CET prices, while subsidies are effective, the relative marginal benefits diminish.

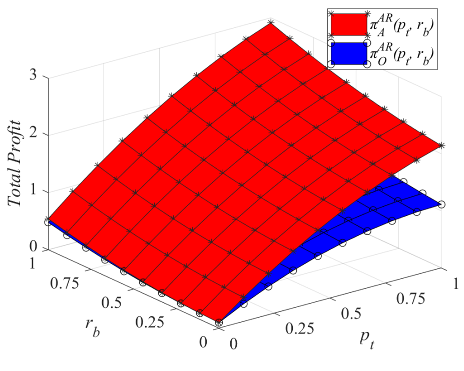


Fig. 7. The impact of p_t and r_b on the total profit of the supply chain under the authorized and outsourcing remanufacturing models

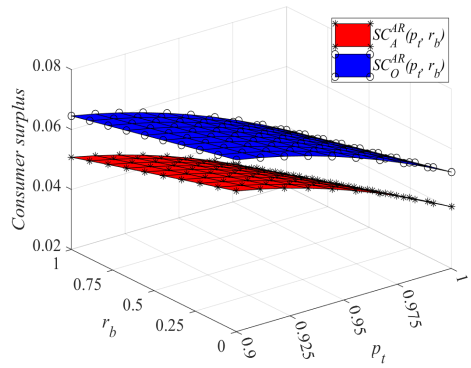


Fig. 8. The impact of p_t and r_b on consumer surplus under authorized and outsourcing remanufacturing models

Fig. 8 shows that under both authorized and outsourced remanufacturing modes, consumer surplus curves show a decreasing trend with the increase in CET prices and remanufacturer subsidies. This indicates that higher CET prices directly increase carbon costs during the remanufacturing process. Enterprises may pass on some or all of the increased costs to consumers, resulting in price increases for remanufactured products and a subsequent reduction in consumer surplus. In the remanufacturing market, government subsidies are often used to reduce enterprise operating costs or incentivize remanufacturing activities. However, the transmission mechanism of subsidies is complex and may cause changes in market structure and adjustments in competitive dynamics. Some enterprises may seize the opportunity to raise prices, ultimately leading to a reduction in consumer surplus.

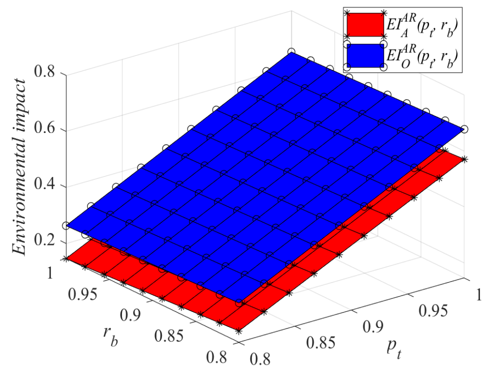


Fig. 9. Environmental impacts of p_t and r_b under authorized and outsourced remanufacturing models

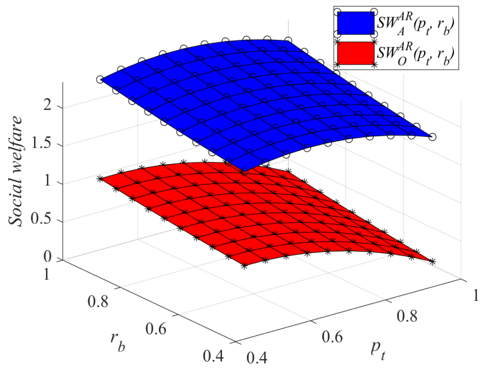


Fig. 10. The impact of p_t and r_b on social welfare under the authorization and outsourcing remanufacturing models

Fig. 9 shows that under both authorized and outsourced remanufacturing modes, environmental impact increases with the increase in CET prices. However, moderate remanufacturer subsidies can alleviate this impact to some extent, mitigating environmental effects. The underlying reason for these phenomena lies mainly in the dual effects of CET prices and remanufacturer subsidies. On the one hand, CET prices directly affect enterprise remanufacturing costs, thereby influencing market prices and demand for remanufactured products. Higher CET prices increase remanufacturing costs, leading enterprises to reduce remanufacturing activities and increase the production of new products, thus exacerbating environmental burdens. On the other hand, remanufacturer subsidies can offset the negative impact of higher CET prices by reducing remanufacturing costs to a certain extent, promoting remanufacturing activities, and thereby reducing environmental impact.

Fig. 10 shows that under the authorized remanufacturing mode, social welfare exhibits an increasing trend followed by a decrease as CET prices increase. This indicates that initially, moderate CET prices contribute to improving social welfare, but excessively high CET prices can have negative effects. With the increase in consumer subsidies, social welfare gradually improves, indicating the positive role of consumer subsidies in stimulating consumption. Under the outsourced remanufacturing mode, the performance of the social welfare function is consistent with the authorized remanufacturing mode. The initial increase in CET prices helps to promote enterprises to adopt more environmentally friendly production methods, thus improving social welfare. However, when CET prices are too high, enterprise production costs significantly increase, resulting in price increases for products and a decrease in consumer demand, ultimately negatively impacting social welfare. The increase in consumer subsidies directly reduces the cost of consumers purchasing remanufactured products, stimulates consumer enthusiasm for purchases, increases market demand, and thereby enhances social welfare. This also reflects the significant role of government subsidy policies in promoting green consumption.

7. Conclusion and future research

In recent years, the importance of the remanufacturing industry has increasingly highlighted due to the dual constraints of global emission reduction pressures and resource-environmental factors. With the implementation of various national carbon emission policies and the promotion of "trade-in" programs, determining the optimal remanufacturing mode under different policy environments has become a pressing issue. However, existing research often focuses on the effects of individual policies, neglecting the combined effects of multiple policies and the impact of consumer replacement behavior. Therefore, it is necessary to conduct a systematic analysis of the combinations of CET policies, remanufacturing subsidies, and the synergistic effects of trade-in programs. This paper's innovation lies in incorporating the "trade-in" program into the discussion of remanufacturing mode selection for the first time, comprehensively considering the impact of consumer replacement behavior on remanufacturing mode decisions, thereby addressing previous research gaps. Additionally, this paper constructs various policy combination models to deeply analyze how different forms of subsidies and CET policies interact to affect OEM remanufacturing mode selection and pricing decisions, providing a more comprehensive decision basis for enterprises and policymakers. This research approach not only expands the dimensions of policy impact research but also offers new theoretical perspectives for the development of the remanufacturing industry.

The key findings of this study are as follows:

(1) Compared to implementing a single policy such as CET, the effect of dual policies (i.e., CET and government subsidies) on promoting remanufacturing is more effective. This finding is consistent with Zhang et al. (2022) and Zhu et al. (2017). Additionally, some important findings are: from a profit perspective, whether the remanufacturing mode is authorized or outsourced, CS policies are optimal only when the subsidy amounts are within a certain threshold. At this point, supply chain

members can achieve mutual benefits. From an environmental perspective, environmental costs are not affected by the choice of government subsidy targets but are influenced by the carbon emissions per unit of new products. When carbon emissions per unit of new products are high, policymakers should consider subsidizing remanufacturers, while if emissions are low, subsidies should be directed to consumers. Moreover, under all three government subsidy policies, consumer surplus in model AC is not always greater than in models AW and AR.

(2) For remanufacturing mode selection, the study finds that without subsidies, the authorized remanufacturing mode provides a greater pricing advantage under low carbon emission costs, enhancing enterprise profit margins. In contrast, under high carbon emission costs or significant market uncertainty, the outsourced remanufacturing mode is a more economical choice. Under CS or RS policies, particularly when subsidy amounts are high, prioritizing the authorized remanufacturing mode maximizes subsidy benefits.

(3) As CET prices and government subsidies rise, there are notable shifts in the total profit of the supply chain, consumer surplus, environmental impact, and social welfare under both the authorized and outsourced remanufacturing modes. Higher levels of CET prices and consumer subsidies result in an increase in overall supply chain profits, regardless of the remanufacturing mode selected. However, the marginal benefits of CET prices and subsidies have certain thresholds. When CET prices are too high, increased enterprise costs lead to a decline in consumer surplus and social welfare, while moderate remanufacturing subsidies can effectively mitigate this negative impact. Additionally, as CET prices and consumer subsidies rise, the authorized remanufacturing mode shows more pronounced environmental and economic advantages, better controlling carbon emissions and production costs, and enhancing social welfare.

Managerial Implications:

(1) Governments should prioritize CS policies: In most cases, CS policies effectively increase market demand for remanufactured products and enterprise profits. When designing subsidy policies, policymakers should consider the threshold amount of subsidies to avoid market price inflation that could offset the subsidy effects. Additionally, supporting trade-in programs through financial incentives can promote the development of a circular economy and reduce waste.

(2) Enterprises should optimize their remanufacturing mode selection: Under no subsidy policies, the authorized remanufacturing mode offers greater pricing advantages under low carbon emission costs, effectively enhancing profit margins. In contrast, under high carbon emission costs or greater market uncertainty, the outsourced remanufacturing mode is more economical. Under consumer or RS policies, especially when subsidies are high, prioritizing the authorized remanufacturing mode can maximize subsidy benefits. Enterprises should dynamically adjust their remanufacturing mode strategies based on market and environmental changes.

(3) Enterprises should improve the quality and technology of remanufactured products: For authorized remanufacturers, enhancing the quality and performance of remanufactured products is a key strategy to expand market share. High-quality remanufactured products can increase consumer acceptance and loyalty, thus boosting sales and profits. Additionally, OEMs should adjust the pricing of new and remanufactured products and increase trade-in discounts under CS policies to enhance market competitiveness.

(4) Strengthen consumer awareness and market marketing: Enterprises should invest in marketing strategies to improve the recognition of remanufactured products, particularly focusing on promoting product quality, reliability, and environmental benefits. Building consumer trust is crucial for enhancing the market acceptance and demand for remanufactured products. Especially under subsidy policies, enterprises should clearly communicate product value and subsidy advantages to maximize market effects.

Furthermore, there are certain constraints within this study that warrant further exploration in future research. Firstly, to simplify the analysis, this study is limited to single-period and single-player scenarios. Future research could extend to multi-period environments, evaluating and comparing the effects of remanufacturing regulations under supply chains with multiple manufacturers, remanufacturers, and retailers. Secondly, this study only considers CET and RS policies; future research could incorporate other environmental policies, such as carbon tax policies. Additionally, investigating how different market structures and consumer behaviors affect policy outcomes, as well as the temporal dynamics and feedback mechanisms of policy implementation, will provide deeper insights for policymakers. These extensions will enrich existing theoretical frameworks and offer more comprehensive empirical evidence for policy formulation, advancing sustainable development goals.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix. Proofs of the main models

Proof of Lemma 1

In the authorized remanufacturing mode without government subsidy (Model AW), according to the sequence of the game, first, the manufacturer determines the licensing fee (f). Next, the manufacturer sets the wholesale price of new products (w_n) and the trade-in rebate offered to the retailer (p_c), while the remanufacturer sets the wholesale price of remanufactured products (wr). Finally, the retailer determines the retail price of new products (p_n), the retail price of remanufactured products (p_r), and the trade-in discount offered to consumers (p_d). Using the backward induction method in game theory, we first consider the retailer's decision problem. The retailer's profit function is $\pi_T^{AW} = (p_n - w_n)d_{nn}^{AW} + (p_n - w_n + p_c - p_d)d_{on}^{AW} + (p_r - w_r)d_{nr}^{AW}$. By differentiating π_T^{AW} with respect to p_n, p_r, and p_d, we obtain equations:

$$\frac{\partial \pi_T^{AW}}{\partial p_n} = \alpha\left(1 - \frac{p_n - p_r}{1-\beta}\right) - \frac{\alpha(p_n - w_n)}{1-\beta} + (1-\alpha)\left(1 - \frac{p_n - p_d}{1-2\delta}\right) - \frac{(p_n - w_n + p_c - p_d)(1-\alpha)}{1-2\delta} + \frac{\alpha(p_r - w_r)}{1-\beta}, \tag{1}$$

$$\frac{\partial \pi_T^{AW}}{\partial p_r} = \frac{\alpha(p_n - w_n)}{1-\beta} + \alpha\left(\frac{p_n - p_r}{1-\beta} - \frac{p_r}{\beta}\right) - (p_r - w_r)\alpha\left(\frac{1}{1-\beta} + \frac{1}{\beta}\right), \tag{2}$$

$$\frac{\partial \pi_T^{AW}}{\partial p_d} = -(1-\alpha)\left(1 - \frac{p_n - p_d}{1-2\delta}\right) + \frac{(p_n - w_n + p_c - p_d)(1-\alpha)}{1-2\delta}. \tag{3}$$

Differentiating Eq. (1), Eq. (2), and Eq. (3) with respect to p_n, p_r, and p_d, we derive the Hessian matrix:

$$H = \begin{bmatrix} \frac{\partial^2 \pi_T^{AW}}{(\partial p_n)^2} & \frac{\partial^2 \pi_T^{AW}}{\partial p_n \partial p_r} & \frac{\partial^2 \pi_T^{AW}}{\partial p_n \partial p_d} \\ \frac{\partial^2 \pi_T^{AW}}{\partial p_r \partial p_n} & \frac{\partial^2 \pi_T^{AW}}{(\partial p_r)^2} & \frac{\partial^2 \pi_T^{AW}}{\partial p_r \partial p_d} \\ \frac{\partial^2 \pi_T^{AW}}{\partial p_d \partial p_n} & \frac{\partial^2 \pi_T^{AW}}{\partial p_d \partial p_r} & \frac{\partial^2 \pi_T^{AW}}{(\partial p_r)^2} \end{bmatrix} = \begin{bmatrix} \frac{2(1-\alpha)\beta + 2(2\alpha\delta - 1)}{(\beta-1)(2\delta-1)} & \frac{2\alpha}{1-\beta} & \frac{2(1-\alpha)}{1-2\delta} \\ \frac{2\alpha}{1-\beta} & \frac{2\alpha}{\beta(\beta-1)} & 0 \\ \frac{2(1-\alpha)}{1-2\delta} & 0 & -\frac{2(1-\alpha)}{1-2\delta} \end{bmatrix}. \tag{4}$$

From Eq. (4), we see that $|H_1(p_n, p_r, p_d)| = \frac{2(1-\alpha)\beta + 2(2\alpha\delta - 1)}{(\beta-1)(2\delta-1)} < 0$, $|H_2(p_n, p_r, p_d)| = \frac{4\alpha(1-2\alpha\delta)}{\beta(\beta-1)(2\delta-1)} > 0$, and $|H_3(p_n, p_r, p_d)| = \frac{8\alpha^2(\alpha-1)}{\beta(\beta-1)(2\delta-1)} < 0$. Therefore, π_T^{AW} is jointly concave with respect to p_n, p_r, and p_d. Setting $\frac{\partial \pi_T^{AW}}{\partial p_n} = 0$, $\frac{\partial \pi_T^{AW}}{\partial p_r} = 0$, and $\frac{\partial \pi_T^{AW}}{\partial p_d} = 0$, we can solve for $p_n = \frac{w_n + 1}{2}$, $p_r = \frac{w_r + \beta}{2}$, and $p_d = \delta + \frac{p_c}{2}$. Substituting p_n and p_r into the demand functions d_{nn}^{AW}, d_{on}^{AW}, and d_{nr}^{AW}, we obtain $d_{nn}^{AW} = \frac{\alpha(\beta - 1 + w_n - w_r)}{2(\beta-1)}$, $d_{on}^{AW} = \frac{(\alpha-1)(1-2\delta-w_n+p_c)}{2(2\delta-1)}$, and $d_{nr}^{AW} = -\frac{\alpha(\beta w_n - w_r)}{2\beta(\beta-1)}$.

Next, we consider the simultaneous decision-making problem of the manufacturer and the remanufacturer. By substituting d_{nn}^{AW}, d_{on}^{AW}, and d_{nr}^{AW} into the manufacturer's profit function, we obtain $\pi_M^{AW} = \frac{(w_n - c_n)\alpha(\beta - 1 + w_n - w_r)}{2(\beta-1)} + \frac{(w_n - c_n - p_c + v)(\alpha-1)(1-2\delta-w_n+p_c)}{2(2\delta-1)} - p_t\left(e\left(\frac{\alpha(\beta-1+w_n-w_r)}{2(\beta-1)} + \frac{(\alpha-1)(1-2\delta-w_n+p_c)}{2(2\delta-1)}\right) - Q\right) - \frac{f\alpha(\beta w_n - w_r)}{2(\beta-1)\beta}$, and differentiating with respect to w_n and p_c obtains equations:

$$\frac{\partial \pi_M^{AW}}{\partial w_n} = \frac{\alpha(\beta-1+w_n-w_r)}{2(\beta-1)} + \frac{\alpha(w_n-c_n)}{2(\beta-1)} + \frac{(\alpha-1)(1-2\delta-w_n+p_c)}{2(2\delta-1)} - \frac{(w_n-c_n-p_c+v)(\alpha-1)}{2(2\delta-1)} - p_t e\left(\frac{\alpha}{2(\beta-1)} - \frac{\alpha-1}{2(2\delta-1)}\right) - \frac{\alpha f}{2(\beta-1)}, \tag{5}$$

$$\frac{\partial \pi_M^{AW}}{\partial p_c} = -\frac{(\alpha-1)(1-2\delta-w_n+p_c)}{2(2\delta-1)} + \frac{(w_n-c_n-p_c+v)(\alpha-1)}{2(2\delta-1)} - \frac{p_t e(\alpha-1)}{2(2\delta-1)}. \tag{6}$$

Similarly, substituting d_{nn}^{AW}, d_{on}^{AW}, and d_{nr}^{AW} into the remanufacturer's profit function obtains $\pi_R^{AW} = -\frac{(w_r - c_n + \Delta - f)\alpha(\beta w_n - w_r)}{2(\beta-1)\beta} + p_t\left(\frac{(e-E)\alpha(\beta w_n - w_r)}{2(\beta-1)\beta} + Q\right)$, and differentiating with respect to w_r obtains:

$$\frac{\partial \pi_R^{AW}}{\partial w_r} = -\frac{\alpha(\beta w_n - w_r)}{2(\beta-1)\beta} + \frac{\alpha(w_r - c_n + \Delta - f)}{2(\beta-1)\beta} - \frac{\alpha p_t(e-E)}{2(\beta-1)\beta}. \tag{7}$$

Differentiating Eq. (5) and Eq. (6) with respect to w_n and p_c, we derive the Hessian matrix：

$$H(w_n, p_c) = \begin{bmatrix} \frac{\partial^2 \pi_R^{AW}}{(\partial w_n)^2} & \frac{\partial^2 \pi_R^{AW}}{\partial p w_n \partial p_c} \\ \frac{\partial^2 \pi_R^{AW}}{\partial p_c \partial w_r} & \frac{\partial^2 \pi_R^{AW}}{(\partial p_c)^2} \end{bmatrix} = \begin{bmatrix} \frac{(2\delta-\beta)\alpha + \beta - 1}{(\beta-1)(2\delta-1)} & \frac{\alpha-1}{2\delta-1} \\ \frac{\alpha-1}{2\delta-1} & -\frac{\alpha-1}{2\delta-1} \end{bmatrix}. \tag{8}$$

From Eq. (8), we see that $|H_1(w_n,p_c)| = \frac{(2\delta-\beta)\alpha+\beta-1}{(\beta-1)(2\delta-1)} < 0$ and $|H_2(w_n,p_c)| = \frac{\alpha(1-\alpha)}{(\beta-1)(2\delta-1)} > 0$. Therefore, π_R^{AW} is jointly concave with respect to w_n and p_c. Setting $\frac{\partial\pi_M^{AW}}{\partial w_n} = 0$, $\frac{\partial\pi_M^{AW}}{\partial p_c} = 0$, and $\frac{\partial\pi_R^{AW}}{\partial w_r} = 0$, we can solve for $w_n = \frac{(E-3e)p_t+\Delta+2\beta-3f-3c_n-2}{\beta-4}$, $w_r = \frac{(2E-\beta e-2e)p_t+\beta^2-\beta(f+c_n+1)+2(\Delta-f-c_n)}{\beta-4}$, and $p_c = \frac{(2E-\beta e)p_t+(2\delta+v-c_n)\beta-2ep_t+2\Delta+3\beta-8\delta-6f-4v-2c_n}{2(\beta-4)}$. Substituting w_n, p_c, and w_r into the demand functions d_{nn}^{AW}, d_{on}^{AW}, and d_{nr}^{AW} and then into the manufacturer's profit function, we obtain $\pi_M^{AW} = -\frac{\alpha\left(\frac{(E-3e)p_t+\Delta+2(\beta-1)-3(f+c_n)}{\beta-4}-c_n\right)((-ep_t-f-c_n+2)\beta+(E+e)p_t+f+\Delta+c_n-2)}{2(\beta-4)(\beta-1)} - \frac{\left(\frac{(E-3e)p_t+\Delta+2(\beta-1)-3(f+c_n)}{\beta-4}-c_n-\frac{-\beta(ep_t+c_n-2\delta-v-3)+2(Ep_t+\Delta-c_n-ep_t)+4v-2(4\delta-3f)}{2(\beta-4)}+v\right)(ep_t+2\delta-v+c_n-1)(\alpha-1)}{4(2\delta-1)} - p_t\left(e\left(-\frac{\alpha((-ep_t-f-c_n+2)\beta+(E+e)p_t+f+\Delta+c_n-2)}{2(\beta-4)(\beta-1)}-\frac{(ep_t+2\delta-v+c_n-1)(\alpha-1)}{4(2\delta-1)}\right)-Q\right) - \frac{\alpha f(\beta^2+((E-2e)p_t-2f+\Delta-2c_n-1)\beta-2(E-e)p_t+2(f-\Delta+c_n))}{2(\beta-4)(\beta-1)\beta}$. Setting $\frac{\partial\pi_M^{AW}}{\partial f} = 0$, we obtain $f = \frac{(-ep_t-c_n+1)\beta^2+8(E-e)p_t-8(c_n-\delta-\beta)}{2(\beta+8)}$. Substituting f into w_n, p_c, and w_r gives $w_n = \frac{(3ep_t+3c_n+1)\beta+2(E+3e)p_t+2(\Delta+3c_n+4)}{2(\beta+8)}$, $p_c = \frac{(2ep_t+2\delta+v+2c_n)\beta+2(E-e)p_t+8(v+2\delta)+2(\Delta-c_n)}{2(\beta+8)}$, and $w_r = \frac{(ep_t+c_n+1)\beta^2+4(ep_t+c_n+2)\beta+4(e-E)p_t+4(c_n-\Delta)}{2(\beta+8)}$. Substituting f, w_n, p_c, and w_r into p_n, p_r, and p_d gives $p_n = \frac{3(ep_t+c_n+1)\beta+2(E+3e)p_t+2(\Delta+3c_n+12)}{4(\beta+8)}$, $p_r = \frac{(ep_t+c_n+3)\beta^2+4(ep_t+c_n+6)\beta+4(e-E)p_t+4(c_n-\Delta)}{4(\beta+8)}$ and $p_d = \frac{(2ep_t+6\delta+v+2c_n)\beta+2(E-e)p_t+8v+2\Delta+48\delta-2c_n}{4(\beta+8)}$. Substituting p_n, p_r, and p_d into d_{nn}^{AW}, d_{on}^{AW}, and d_{nr}^{AW} yields $d_{nn}^{AW} = \frac{\alpha(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+2p_t(3E+e)+6\Delta+2c_n-8)}{4(\beta+8)(\beta-1)}$, $d_{on}^{AW} = \frac{-(ep_t+2\delta-v+c_n-1)(\alpha-1)}{4(2\delta-1)}$, and $d_{nr}^{AW} = \frac{-\alpha((ep_t+c_n)\beta+p_t(E-e)+\Delta-c_n)(\beta+2)}{2(\beta+8)(\beta-1)\beta}$. Therefore, $\pi_M^{AW} = \frac{\alpha((3ep_t+c_n+1)\beta+(2E+6e)p_t+2\Delta-10c_n+8)(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+2p_t(3E+e)+6\Delta+2c_n-8)}{8(\beta+8)^2(\beta-1)} + \frac{(\alpha-1)(-ep_t+2\delta-v+c_n-1)(ep_t+2\delta-v+c_n-1)}{8(2\delta-1)} - p_t\left(e\left(\frac{\alpha(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+2p_t(3E+e)+6\Delta+2c_n-8)}{4(\beta+8)(\beta-1)}-\frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)}{4(2\delta-1)}\right)-Q\right) - \frac{(\beta+2)((-ep_t-c_n+1)\beta^2+8(E-e)p_t-8c_n+8\Delta+8\beta)\alpha((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n)}{4(\beta+8)^2(\beta-1)\beta}$, $\pi_R^{AW} = \frac{-4\left(\frac{1}{2}(ep_t+c_n+1)\beta^2+\left(\left(E+\frac{e}{2}\right)p_t+\frac{c_n}{2}-\frac{v}{4}+\Delta-\frac{3\delta}{2}+4\right)\beta+\frac{11}{2}(E-e)p_t-\frac{11c_n}{2}-2v+\frac{11\Delta}{2}-12\delta\right)((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n)(\beta+2)\alpha+8Q\beta p_t(\beta-1)(\beta+8)^2}{8(\beta+8)^2(\beta-1)\beta}$, and $\pi_T^{AW} = \frac{(-\beta^2(ep_t+c_n-1)-\beta(ep_t+c_n-7)+2p_t(3E+e)+6\Delta+2c_n-8)((-3ep_t-3c_n+1)\beta+(-2E-6e)p_t-2\Delta-6c_n+8)\alpha}{16(\beta+8)^2(\beta-1)} - \frac{(\alpha-1)(ep_t+2\delta-v+c_n-1)(v-c_n-2\delta-ep_t)}{16(2\delta-1)} - \frac{(\beta+2)((-ep_t-c_n+1)\beta^2-4(ep_t+c_n-2)\beta+4(E-e)p_t+4\Delta-4c_n)\alpha((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n)}{8(\beta+8)^2(\beta-1)\beta}$.

Proof of Lemmas 2-6: Following a similar process to the proof of Lemma 1, we can derive the results of Lemmas 2-6.

Proof of Theorem 1:

Proposition 5(i): First, under the authorized remanufacturing model, comparing the optimal pricing decisions between no subsidy (AW) and consumer subsidy (AC), we define $\Delta p_{n_1} = p_n^{AW} - p_n^{AC}$. If $r_a > r_{a_1} = \frac{-6(\beta-1)\left(\frac{1}{2}(5ep_t+5c_n-3)\beta^2+((E+7e)p_t+\Delta+7c_n-12)\beta-4(E-e)p_t-4(\Delta-c_n)\right)}{20\beta^2+176\beta+128}$, then $\Delta p_{n_1} < 0$. Defining $\Delta p_{r_1} = p_r^{AW} - p_r^{AC}$, if $r_a > r_{a_2} = \frac{-2(\beta-1)(\beta+2)\left(\frac{1}{2}(5ep_t+5c_n-3)\beta^2+((E+7e)p_t+\Delta+7c_n-12)\beta-4(E-e)p_t-4(\Delta-c_n)\right)}{20\beta^2+176\beta+128}$, then $\Delta p_{r_1} < 0$. Defining $\Delta w_{n_1} = w_n^{AW} - w_n^{AC}$, if $r_a > r_{a_3} = 2r_{a_1} = \frac{-6(\beta-1)\left(\frac{1}{2}(5ep_t+5c_n-3)\beta^2+((E+7e)p_t+\Delta+7c_n-12)\beta-4(E-e)p_t-4(\Delta-c_n)\right)}{10\beta^2+88\beta+64}$, then $\Delta w_{n_1} < 0$. Defining $\Delta w_{r_1} = w_r^{AW} - w_r^{AC}$, if $r_a > r_{a_4} = 2r_{a_2} = \frac{-2(\beta-1)(\beta+2)\left(\frac{1}{2}(5ep_t+5c_n-3)\beta^2+((E+7e)p_t+\Delta+7c_n-12)\beta-4(E-e)p_t-4(\Delta-c_n)\right)}{10\beta^2+88\beta+64}$, then $\Delta w_{r_1} < 0$. Defining $\Delta p_{c_1} = p_c^{AW} - p_c^{AC}$, if $r_a > r_{a_5} = \frac{-6(\beta-1)\left(\frac{1}{2}(5ep_t+5c_n+5\delta-7)\beta^2+\left((E+7e)p_t+\Delta+7c_n+\frac{44\delta}{3}-\frac{58}{3}\right)\beta-4(E-e)p_t-4(\Delta-c_n)+\frac{32\delta}{3}-\frac{16}{3}\right)}{10\beta^2+88\beta+64}$, then $\Delta p_{n_1} < 0$. Defining $\Delta p_{d_1} = p_d^{AW} - p_d^{AC}$, if $r_a > r_{a_6} = \frac{-6(\beta-1)\left(\frac{1}{2}(5ep_t+5c_n+10\delta-8)\beta^2+((E+7e)p_t+\Delta+7c_n+44\delta-34)\beta-4(E-e)p_t-4(\Delta-c_n)+32\delta-16\right)}{20\beta^2+176\beta+128}$, then $\Delta p_{d_1} < 0$. Comparing the sizes of r_{a_1}, r_{a_2}, r_{a_3}, r_{a_4}, r_{a_5} and r_{a_6}, we obtain $r_{a_3} > r_{a_4} > r_{a_1} > r_{a_2}$, $r_{a_3} > r_{a_5}$, and $r_{a_3} > r_{a_6}$. Therefore, when $r_a = r_{a_3} = r_a^A > \frac{-6(\beta-1)\left(\frac{1}{2}(5ep_t+5c_n-3)\beta^2+((E+7e)p_t+\Delta+7c_n-12)\beta-4(E-e)p_t-4(\Delta-c_n)\right)}{10\beta^2+88\beta+64}$, $w_n^{AW} < w_n^{AC}$, $w_r^{AW} < w_r^{AC}$, $p_n^{AW} < p_n^{AC}$, $p_r^{AW} < p_r^{AC}$, $p_c^{AW} < p_c^{AC}$, $p_d^{AW} < p_d^{AC}$. Following a similar process to the proof of Proposition 1(i), we can derive the results of Propositions 1(ii) and 1(iii), where $e_1^A = \frac{-2((\beta+8)r_a+(\beta-1)r_b)}{3(\beta-1)p_t(\beta+2)} + \frac{(-5c_n+3)\beta^2+(-2Ep_t-\Delta-14c_n+24)\beta+8(Ep_t+\Delta-c_n)}{(5\beta+4)p_t(\beta+2)}$.

Proof of Theorem 2:

(i) For the relationship between π_M^{AW} and π_M^{AR}, we have:

$\pi_M^{AW} - \pi_M^{AR} = \frac{\alpha((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n+r_b)r_b}{\beta(\beta+8)(\beta-1)} < 0$, when $\frac{\sqrt{\beta(ep_t+c_n-1)^2(\beta+8)(\beta-1)}}{2} + \frac{(2\beta e-2E+2e)p_t}{2} + \frac{2(1-\beta)c_n}{2} - \Delta = r_{b_1}^A < r_b < r_{b_2}^A = -\frac{\sqrt{\beta(ep_t+c_n-1)^2(\beta+8)(\beta-1)}}{2} + \frac{(-2\beta e-2E+2e)p_t}{2} + \frac{2(1-\beta)c_n}{2} - \Delta$, $\pi_M^{AW} < \pi_M^{\mathrm{AR}}$.

For the relationship between π_M^{AW} and π_M^{AC} and the relationship between π_M^{AC} and π_M^{AR}, we have:

when $\frac{\begin{pmatrix}(7E^2+15Ee+24e^2)p_t^2+((-11E+48e)c_n+(14E+15e)\Delta-27E+(-8r_b-75)e)p_t \\ +24c_n^2+(-11\Delta-8r_b-89)c_n+7\Delta^2-27\Delta-3r_a+64\end{pmatrix}\beta^2 + \begin{pmatrix}(-12E^2-12Ee-4e^2)p_t^2+((-52E-8e)c_n+(-24E-12e)\Delta+(-4r_a-8r_b+12)E+(-56r_b+84)e)p_t- \\ 4c_n^2+(-52\Delta-56r_b+124)c_n-12\Delta^2+(-8r_b+12)\Delta-4r_b^2-68\end{pmatrix}\beta - 32((E-e)p_t+\Delta-c_n+r_b)^2}{8(\beta+8)^2(\beta-1)\beta} = r_{a_1}^A < r_a < r_{a_2}^A = \frac{(1-\beta)\left(\left(\frac{5ep_t}{2}+\frac{5c_n}{2}-\frac{3}{2}\right)\beta^2+((E+7e)p_t+\Delta+7c_n-12)\beta+(-4E+4e)p_t-4\Delta+4c_n\right)^2}{5\left(\beta+\frac{4}{5}\right)(\beta+8)\beta(Ep_t+\Delta+\beta-1)}$, $\pi_M^{AW} - \pi_M^{AC} < 0, \pi_M^{AR} - \pi_M^{AC} < 0$.

(ii) For the relationship between π_R^{AW} and π_R^{AR}, we have:

$$\pi_R^{AW} - \pi_R^{AR} = -4\left(\left(e\frac{p_t}{2}+\frac{c_n}{2}+\frac{1}{2}\right)\beta^2 + \left(\left(E+\frac{e}{2}\right)p_t + \frac{p_t}{2} - \frac{v}{4} + \Delta - \frac{3\delta}{2} + 4\right)\beta + \left(\frac{11E}{2}-\frac{11e}{2}\right)p_t - \frac{11p_t}{2} - 2v + \frac{11\Delta}{2} - 12\delta\right)\left((ep_t+c_n)\beta + (E-e)p_t + \Delta - c_n\right)(\beta+2)\alpha + 6Q\beta p_t(\beta-1)(\beta+8)^2 + \left((ep_t+c_n)\beta + (E-e)p_t + \Delta - c_n + r_b\right)^2(\beta+2)^2\alpha,$$

when $-\sqrt{4\alpha\begin{pmatrix}\begin{pmatrix}\left(e\frac{p_t}{2}+\frac{c_n}{2}+\frac{1}{2}\right)\beta^2+\left(\left(E+\frac{e}{2}\right)p_t+\frac{c_n}{2}-\frac{v}{4}+\Delta-\frac{3\delta}{2}+4\right)\beta+ \\ \left(\frac{11E}{2}-\frac{11e}{2}\right)p_t-\frac{11c_n}{2}-2v+\frac{11\Delta}{2}-12\delta\end{pmatrix} \\ \left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n\right)(\beta+2)\alpha-\frac{3Q\beta p_t(\beta-1)(\beta+8)^2}{2}\end{pmatrix}} + ((ep_t+c_n)\beta^2 + ((E+e)p_t+\Delta+c_n)\beta + (2E-2e)p_t + 2\Delta - 2c_n)\alpha = r_{b_3}^A < r_b < r_{b_4}^A = \sqrt{4\alpha\begin{pmatrix}\begin{pmatrix}\left(e\frac{p_t}{2}+\frac{c_n}{2}+\frac{1}{2}\right)\beta^2+\left(\left(E+\frac{e}{2}\right)p_t+\frac{c_n}{2}-\frac{v}{4}+\Delta-\frac{3\delta}{2}+4\right)\beta+ \\ \left(\frac{11E}{2}-\frac{11e}{2}\right)p_t-\frac{11c_n}{2}-2v+\frac{11\Delta}{2}-12\delta\end{pmatrix} \\ \left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n\right)(\beta+2)\alpha-\frac{3Q\beta p_t(\beta-1)(\beta+8)^2}{2}\end{pmatrix}} + ((ep_t+c_n)\beta^2 + ((E+e)p_t+\Delta+c_n)\beta + (2E-2e)p_t + 2\Delta - 2c_n)\alpha$, $\pi_R^{AW} - \pi_R^{AR} < 0$.

For the relationship between π_R^{AW} and π_R^{AC} and the relationship between π_R^{AC} and π_R^{AR}, we have:

When $\frac{(-5ep_t-5c_n-3)\beta^3+((-8E-9e)p_t-8\Delta-9c_n-21)\beta^2+((-38E+6e)p_t-38\Delta+6c_n+24)\beta+(-8E+8e)p_t-8\Delta+8c_n}{5\beta^2+14\beta+8} = r_{a_3}^A < r_b < r_{a_4}^A = \frac{2\left(\left(\frac{5ep_t}{2}+\frac{5c_n}{2}-\frac{3}{2}\right)\beta^2+((E+7e)p_t+\Delta+7c_n-12)\beta-4(E-e)p_t-4\Delta+4c_n\right)(1-\beta)}{5\beta^2+14\beta+8}$, $\pi_R^{AC} - \pi_R^{AW} > 0, \pi_R^{AC} - \pi_R^{AR} > 0$.

(iii) For the relationship between π_T^{AW} and π_T^{AR}, we have:

$$\pi_T^{AW} - \pi_T^{AR} = \frac{\alpha r_b(-\beta^3(ep_t+c_n+1)+3\beta^2(ep_t+c_n)+7\beta^2+10E\beta p_t+6\beta ep_t+10\Delta\beta+8Ep_t+6\beta c_n+5\beta r_b-8ep_t+8\Delta-8\beta-8c_n+4r_b)}{4(\beta+8)^2(\beta-1)\beta},$$

when $\frac{-\sqrt{\beta(ep_t+c_n-1)^2(\beta-1)(\beta+8)^2(\beta+2)^2}+(ep_t+c_n-1)\beta^3+(-3ep_t-3c_n-7)\beta^2+((-10E-6e)p_t-10\Delta-6c_n+8)\beta-8(E-e)p_t-8\Delta+8c_n}{10\beta+8} = r_{b_5}^A <$ $r_b < r_{b_6}^A = \frac{\sqrt{\beta(ep_t+c_n-1)^2(\beta-1)(\beta+8)^2(\beta+2)^2}+(ep_t+c_n-1)\beta^3+(-3ep_t-3c_n-7)\beta^2+((-10E-6e)p_t-10\Delta-6c_n+8)\beta-8(E-e)p_t-8\Delta+8c_n}{10\beta+8}$, $\pi_T^{AW} - \pi_T^{AR} < 0$.

For the relationship between π_T^{AW} and π_T^{AR} and the relationship between π_T^{AC} and π_T^{AR}, we have:

When $\frac{-\sqrt{-12\alpha(\beta+2)\left(-\frac{\alpha(\beta+2)(\beta+8)^2(ep_t+c_n-1)^2}{12}+(Ep_t+\Delta+\beta-1)^2\right)}+(\beta^2(ep_t+c_n-1)+(-8ep_t-8c_n-4)\beta+(-12E-20e)p_t-12\Delta-20c_n+32)\alpha}{12\alpha} = r_{a_5}^A <$ $r_b < r_{a_6}^A = \frac{\sqrt{-12\alpha(\beta+2)\left(-\frac{\alpha(\beta+2)(\beta+8)^2(ep_t+c_n-1)^2}{12}+(Ep_t+\Delta+\beta-1)^2\right)}+(\beta^2(ep_t+c_n-1)+(-8ep_t-8c_n-4)\beta+(-12E-20e)p_t-12\Delta-20c_n+32)\alpha}{12\alpha}$, $\pi_T^{AC} - \pi_T^{AW} > 0, \pi_T^{AC} - \pi_T^{AR} > 0$.

Proof of Theorem 3:

$$CS^{AW} = \alpha \int_{\frac{p_n^{AW}-p_r^{AW}}{1-\beta}}^{1} (\theta - p_n^{AW}) d\theta + \alpha \int_{\frac{p_r^{AW}}{\beta}}^{\frac{p_n^{AW}-p_r^{AW}}{1-\beta}} (\beta\theta - p_r^{AW}) d\theta + (1-\alpha) \int_{\frac{p_n^{AW}-p_d^{AW}}{1-2\delta}}^{1} ((1-\delta)\theta - p_n^{AW} + p_d^{AW}) d\theta,$$

$$CS^{AR} = \alpha \int_{\frac{p_n^{AR}-p_r^{AR}}{1-\beta}}^{1} (\theta - p_n^{AR}) d\theta + \alpha \int_{\frac{p_r^{AR}}{\beta}}^{\frac{p_n^{AR}-p_r^{AR}}{1-\beta}} (\beta\theta - p_r^{AR}) d\theta + (1-\alpha) \int_{\frac{p_n^{AR}-p_d^{AR}}{1-2\delta}}^{1} ((1-\delta)\theta - p_n^{AR} + p_d^{AR}) d\theta,$$

$$CS^{AC} = \alpha \int_{\frac{p_n^{AC}-p_r^{AC}}{1-\beta}}^{1} (\theta - p_n^{AC}) d\theta + \alpha \int_{\frac{p_r^{AC}}{\beta}}^{\frac{p_n^{AC}-p_r^{AC}}{1-\beta}} (\beta\theta - p_r^{AC}) d\theta + (1-\alpha) \int_{\frac{p_n^{AC}-p_d^{AC}}{1-2\delta}}^{1} ((1-\delta)\theta - p_n^{AC} + p_d^{AC}) d\theta,$$

According to theorem 1(ii), $p_n^{AW} < p_n^{AR}$, $p_c^{AW} < p_c^{AR}$, $p_d^{AW} < p_d^{AR}$, $p_r^{AW} > p_r^{AR}$. Hence, $CS^{AW} < CS^{AR}$.

When $r_b = r_{b_7}^A > \frac{(ep_t+c_n)\beta^2}{6} + \frac{((E+e)p_t+\Delta+c_n-6)\beta}{6} + \frac{(-4E-2e)p_t}{6} - \frac{2\Delta}{3} - \frac{c_n}{3} + 1$, $CS^{AR} < CS^{AC}$.

Proof of Theorem 4:

$EI^{AW} - EI^{AR} = -\frac{\alpha r_b((E+2e)\beta+2E-2e)}{2(\beta+8)(\beta-1)\beta}$, when $e > e_2^A = \frac{E(\beta+2)}{2(1-\beta)}$, $EI^{AW} - EI^{AR} < 0$. When $E \le e < e_3^A = \frac{(-\beta^2 c_n + (-Ep_t-\Delta-c_n-r_b)\beta-2Ep_t+v-2\Delta+c_n-2r_b)\alpha - v + c_n}{(-1+(\beta^2+\beta-1)\alpha)p_t}$, $EI^{AR} - EI^{AC} < 0$. In summary, when $e_2^A \le e < e_3^A$, $EI^{AW} < EI^{AR} < EI^{AC}$. When $e \ge e_3^A$, $EI^{AW} < EI^{AC} < EI^{AR}$.

Proof of Theorem 5:

Following a similar process for proving theorem 1, we can obtain the result of theorem 5.

Proof of Theorem 6:

Following a similar process for proving theorem 2, we can get the result of theorem 6. The related thresholds are as follows:

$r_{b_1}^O = (-\beta e - E + e)p_t + (1-\beta)c_n - \Delta$,

$$\frac{-\sqrt{(\beta-1)\begin{pmatrix} p_t^2(\beta-1)(\beta+1)^2e^2-2((-3\beta^2+2\beta+1)c_n+(-3\beta-1)r_b+(\beta-1)(Ep_t+\Delta+4\beta))p_t(\beta+1)e+(9\beta^3-3\beta^2-5\beta-1)c_n^2 \\ -6(Ep_t+\Delta+4\beta+r_b)(\beta-1)\left(\beta+\frac{1}{3}\right)c_n+(-3\beta-1)r_b^2-6(Ep_t+\Delta)\left(\beta+\frac{1}{3}\right)r_b+(\beta-1)(Ep_t+\Delta+4\beta)^2 \end{pmatrix}} +(-ep_t-3c_n+4)\beta^2+(Ep_t+\Delta+2c_n-4)\beta+(e-E)p_t-\Delta+c_n}{3\beta+1} = r_{a_1}^O < r_a < r_{a_2}^O =$$

$$\frac{\sqrt{(\beta-1)\begin{pmatrix} p_t^2(\beta-1)(\beta+1)^2e^2-2((-3\beta^2+2\beta+1)c_n+(-3\beta-1)r_b+(\beta-1)(Ep_t+\Delta+4\beta))p_t(\beta+1)e+(9\beta^3-3\beta^2-5\beta-1)c_n^2 \\ -6(Ep_t+\Delta+4\beta+r_b)(\beta-1)\left(\beta+\frac{1}{3}\right)c_n+(-3\beta-1)r_b^2-6(Ep_t+\Delta)\left(\beta+\frac{1}{3}\right)r_b+(\beta-1)(Ep_t+\Delta+4\beta)^2 \end{pmatrix}} +(-ep_t-3c_n+4)\beta^2+(Ep_t+\Delta+2c_n-4)\beta+(e-E)p_t-\Delta+c_n}{3\beta+1}, \quad r_{a_3}^O =$$

$$\frac{\sqrt{\left(\begin{array}{c}\left(-5e^2{p_t}^2-6\left(v-2\delta-c_n+\frac{1}{3}\right)ep_t+15{c_n}^2+(-6v+12\delta-42)c_n+12v-24\delta+28\right)\beta^3+\\ \left((4Ee+5e^2){p_t}^2+\left((-12E+2e)c_n+(4v+4\Delta-8\delta-4)e+6\left(v-2\delta+\frac{7}{3}\right)E\right)p_t-7{c_n}^2+(4v-12\Delta-8\delta+28)c_n+(6v-12\delta+14)\Delta-8v+16\delta-24\right)\beta^2\\ +\left((E+e)^2{p_t}^2+\left((2E-6e)c_n+(2v+2\Delta-4\delta+6)e+2E(v+\Delta-2\delta-3)\right)p_t-7{c_n}^2+(2v+2\Delta-4\delta+14)c_n+\Delta^2+(2v-4\delta-6)\Delta-4v+8\delta-4\right)\beta\\ -\left((E-e)p_t+\Delta-c_n\right)^2\end{array}\right)(\beta-1)}\\ +(-ep_t-3c_n+4)\beta^2+(Ep_t+\Delta+2c_n-4)\beta+(e-E)p_t-\Delta+c_n}{3\beta+1},\ r_{b_2}^O=$$

$-2(\beta e+E-e)p_t+2(1-\beta)c_n-2\Delta,\ r_{a_4}^O=(-2\beta e-2E+2e)p_t-2(\beta-1)c_n-2\Delta-r_b,\ r_{a_5}^O=(-2\beta e-2E+2e)p_t-2(\beta-1)c_n-2\Delta,\ r_{b_3}^O=\sqrt{-\left((\beta e+E-e)p_t+\beta c_n+\Delta-c_n\right)\left((-3\beta e+E-e)p_t+(-3c_n+4)\beta+\Delta-c_n\right)}+(-\beta e-E+e)p_t+$

$$(1-\beta)c_n-\Delta,\ r_{a_6}^O=\frac{\sqrt{(\beta-1)\left(\begin{array}{c}9\left(ep_t+c_n-\frac{4}{3}\right)^2\beta^3+\left((-6Ee-3e^2){p_t}^2+\left(-6(E+e)c_n+(-6\Delta-6r_b+16)e+8E\right)p_t-3{c_n}^2+(-6\Delta-6r_b+16)c_n+8\Delta-16\right)\beta^2+\\ \left((E+5e)(E-e){p_t}^2+\left((4E-10e)c_n+4(\Delta+r_b+2)e+2E(\Delta-3r_b-4)\right)p_t-5{c_n}^2+(4\Delta+4r_b+8)c_n+\Delta^2+(-6r_b-8)\Delta-3{r_b}^2\right)\beta\\ -\left((E-e)p_t+\Delta-c_n+r_b\right)^2\end{array}\right)}\\ +(-3ep_t-3c_n+4)\beta^2+\left((E+2e)p_t+\Delta+2c_n-4\right)\beta+(e-E)p_t-\Delta+c_n}{3\beta+1},\ r_{a_7}^O=$$

$$\frac{-\sqrt{(\beta-1)\left(\begin{array}{c}9\left(ep_t+c_n-\frac{4}{3}\right)^2\beta^3+\left((-6Ee-3e^2){p_t}^2+\left(-6(E+e)c_n+(-6\Delta-6r_b+16)e+8E\right)p_t-3{c_n}^2+(-6\Delta-6r_b+16)c_n+8\Delta-16\right)\beta^2+\\ \left((E+5e)(E-e){p_t}^2+\left((4E-10e)c_n+4(\Delta+r_b+2)e+2E(\Delta-3r_b-4)\right)p_t-5{c_n}^2+(4\Delta+4r_b+8)c_n+\Delta^2+(-6r_b-8)\Delta-3{r_b}^2\right)\beta\\ -\left((E-e)p_t+\Delta-c_n+r_b\right)^2\end{array}\right)}\\ +(-3ep_t-3c_n+4)\beta^2+\left((E+2e)p_t+\Delta+2c_n-4\right)\beta+(e-E)p_t-\Delta+c_n}{3\beta+1},\ r_{a_8}^O=$$

$$\frac{\sqrt{7}\sqrt{\left(\begin{array}{c}\left(\frac{3e^2{p_t}^2}{7}+\frac{6e(c_n-2)p_t}{7}+\frac{3{c_n}^2}{7}-\frac{12c_n}{7}+\frac{16}{7}\right)\beta^3+\left(-6\left(\frac{5e}{6}+E\right)e\frac{{p_t}^2}{7}+\left(\left(-\frac{6E}{7}-\frac{10e}{7}\right)c_n+\left(\frac{8}{7}-\frac{6\Delta}{7}\right)e+\frac{20E}{7}\right)p_t-\frac{5{c_n}^2}{7}+\left(\frac{8}{7}-\frac{6\Delta}{7}\right)c_n+\frac{20\Delta}{7}-\frac{16}{7}\right)\beta^2\\ +\left(\left(E-\frac{e}{7}\right)p_t+\Delta-\frac{c_n}{7}-\frac{4}{7}\right)\left((E-e)p_t+\Delta-c_n\right)\beta+\frac{\left((E-e)p_t+\Delta-c_n\right)^2}{7}\end{array}\right)(\beta-1)}\\ +(-3ep_t-3c_n+4)\beta^2+\left((E+2e)p_t+\Delta+2c_n-4\right)\beta+(e-E)p_t-\Delta+c_n}{3\beta+1}.$$

Proof of Theorem 7:

Following a similar process for proving theorem 3, we can get the result of theorem 7.

Proof of Theorem 8:

Following a similar process for proving theorem 4, we can get the result of theorem 8.

Proof of Theorem 9:

Following a similar process for proving theorem 1, we can obtain the result of theorem 9.

Proof of Theorem 10:

Following a similar process for proving theorem 2, we can obtain the result of theorem 10. The related thresholds are as follows:

$$e_1^W=\frac{\sqrt{\begin{array}{c}36\Delta^2+\left((-8\beta^2+56\beta-48)v+72p_tE+16\left((\beta-6)\delta+\left(\frac{\beta}{2}-\frac{9}{2}\right)c_n-\frac{\beta}{2}+9\right)(\beta-1)\right)\Delta+16(\beta-1)^2v^2-8(\beta-1)\\ \left(E(\beta-6)p_t-(-8\delta+(\beta-14)c_n-2\beta+20)(\beta-1)\right)v+36E^2{p_t}^2+16E\left((\beta-6)\delta+\left(\frac{\beta}{2}-\frac{9}{2}\right)c_n-\frac{\beta}{2}+9\right)(\beta-1)p_t+(\beta-1)^2\\ \left(64\delta^2+\left((-16\beta+224)c_n+32\beta-320\right)\delta+(\beta-14)^2{c_n}^2+(-2\beta^2+64\beta-464)c_n+\beta^2-36\beta+304\right)\end{array}}\\ +(4v-8\delta-2c_n-4)\beta-6p_tE-4v-6\Delta+8\delta+2c_n+4}{(\beta-1)(\beta-12)p_t},$$

$$e_2^W=\frac{(-3c_n-4)\beta^3+(-7Ep_t-7\Delta+12\delta+2v+3c_n-40)\beta^2+(-44Ep_t-44\Delta+120\delta+20v+84c_n-64)\beta-24p_tE+32v-24\Delta+192\delta+24c_n}{3p_t(\beta^3-\beta^2-28\beta-8)},$$

$$e_3^W=\frac{(-19c_n+20)\beta^3+(Ep_t+\Delta-81c_n+176)\beta^2+(96Ep_t+96\Delta-96c_n+128)\beta+128p_tE+128\Delta-128c_n}{p_t(19\beta^3+81\beta^2+96\beta+128)},$$

$$r_{a_1}=-\frac{6\sqrt{\beta^2(\beta-1)(\beta-8)(Ep_t+\Delta+\beta-1)^2}+(-15ep_t-15c_n+12)\beta^3+\left((-3E-2e)p_t-3\Delta-2c_n+12\right)\beta^2\\ +\left((7E+13e)p_t+7\Delta+13c_n-24\right)\beta-4(E-e)p_t-4\Delta+4c_n}{15\beta^2+17\beta+4},$$

$$r_{a_2}=\frac{6\sqrt{\beta^2(\beta-1)(\beta-8)(Ep_t+\Delta+\beta-1)^2}+(-15ep_t-15c_n+12)\beta^3+\left((-3E-2e)p_t-3\Delta-2c_n+12\right)\beta^2\\ +\left((7E+13e)p_t+7\Delta+13c_n-24\right)\beta-4(E-e)p_t-4\Delta+4c_n}{15\beta^2+17\beta+4},$$

$$r_{a_3} = -\frac{6\sqrt{2}\beta(Ep_t+\Delta+\beta-1)-5\left(\beta+\frac{4}{5}\right)\left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n\right)}{5\beta+4},$$

$$r_{a_4} = \frac{6\sqrt{2}\beta(Ep_t+\Delta+\beta-1)-5\left(\beta+\frac{4}{5}\right)\left((ep_t+c_n)\beta+(E-e)p_t+\Delta-c_n\right)}{5\beta+4},$$

$$r_{a_5} = \frac{-2\sqrt{-7\beta^2(5\beta^2-\beta-4)(Ep_t+\Delta+\beta-1)^2}+5(\beta-1)\left(\beta+\frac{4}{5}\right)\left((-3ep_t-3c_n+4)\beta+(E-e)p_t+\Delta-c_n\right)}{15\beta^2+17\beta+4},$$

$$r_{a_6} = \frac{2\sqrt{-7\beta^2(5\beta^2-\beta-4)(Ep_t+\Delta+\beta-1)^2}+5(\beta-1)\left(\beta+\frac{4}{5}\right)\left((-3ep_t-3c_n+4)\beta+(E-e)p_t+\Delta-c_n\right)}{15\beta^2+17\beta+4},$$

$$r_{b_1} = (-\beta e - E + e)p_t + (1-\beta)c_n - \Delta,$$

$$r_{b_2} = \frac{(-17ep_t-17c_n+16)\beta^2+\left((-E-47e)p_t-\Delta-47c_n+112\right)\beta+(64E+64e)p_t+64\Delta+64c_n-128}{\beta-64}.$$