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Mathematical modelling of torsional vibrations of a truncated conical shell located in an elastic medium

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In this article, mathematical modelling of torsional vibrations of a truncated circular conical shell located in an elastic medium is carried out. It is believed that the shell is exposed to external dynamic loads, and its material is homogeneous and isotropic. Based on the exact mathematical formulation of the problem, general equations of nonstationary torsional vibrations of a truncated conical shells have been developed, from which, in particular cases, the equations of vibration of a truncated conical rod, as well as a circular cylindrical shell and a round rod follow. The desired functions found by solving the vibrational equations are used to construct a method for computing the stress-strain state of any point in the system under investigation. As an example, the problem of propagation of harmonic torsional waves in a truncated conical rod is solved. The effects of the interacting medium, the angle of attack and other physic-mechanical parameters of the rod material on the "frequency-wave number" dependence are estimated.

1. Introduction

In many areas of science and technology, conical shell structures are well used, which puts forward studying of vibratory processes in such systems to ensure the safety and stability. They are one of the main structural parts of various aircraft, submarines, protective tanks and reservoirs (Khudoynazarov, 2024; Sofiyev, 2014). In the case of shell points deformations are small-scale compared to its thickness, then calculations and analysis of shell structures were carried out based on the linear theory (Filippov & Kudainazarov, 1998; Khudoynazarov & Yalgashev, 2021). In certain practical applications of vibrating systems, to gain better exactness it is required to apply the nonlinear theory (Alijani & Amabili, 2014; Sofiyev, 2012). This approach investigates nonlinear vibrations of TCS, including those constructed from functionally graded material (FGM).

Nonstationary vibrations of truncated conical shells (TCS) receive significantly less focus than vibrations of cylindrical shells and rods (Khudoynazarov et al., 2022). In an article by Khudoynazarov and Ismoilov (2023) a model of nonstationary TV of a truncated conical elastic layer of arbitrary thickness located in a deformable medium is proposed. Refined equations of torsional vibrations (TV) of a truncated conical layer, the material of which is assumed homogeneous and isotropic, are derived. Developed an algorithm that enables the unique determination of the stress-strain state at any location within the cross-section of the layer under consideration by spatial and temporal coordinates utilizing the field of the desired functions. Additionally, the research included various limiting and special cases derived from these results.

The study focuses on the investigation of forced vibration in layered composite conical shells in layered composite conical shells. The higher –order shear deformation theory is employed, which takes into account rotational inertia and geometric nonlinearity in all kinematic parameters (Marco & Prabakaran, 2020).

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This paper focuses on the use of the generalized coordinate method to derive the equation of movement and develop the amplitude equation for nonlinear vibrations of conical shells (Bakhtiari & Lakis, 2020). Have been found solutions of the equation of amplitude for isotropic materials in three different shell theories. It made it possible to find differences and similarities of simplifying assumptions of these theories. In addition, research is done on the resonance analysis and harmonic vibration of FGM TCS under harmonic excitation (Aris & Ahmadi, 2020). Using the theory of Karman–Donnell and huge strain theory, the governing equations for FGM TCS are derived. The several scaling approach is used to determine the principal resonance and the effects of cone angle and excitation amplitude variations are investigated using simulations.

In the research work (Shekari et al., 2017) developed a higher order theory to explore the free oscillation of turn around truncated multilayer conical shells. The conditions of continuous displacement at the interface were taken into account. When studying the dynamic behaviour of shells and rods, one of the main problems is the choice of vibration equations. In most cases, researchers, based on the certain physical and mechanical characteristics of the material, develop the necessary vibration equations using various methods (Kushnarenko & Beridze, 2000; Beridze, 1999). Using general solutions in transformations of three-dimensional elasticity theory issues is one of these techniques (Khudoynazarov, 2003; Khalmuradov et al., 2022). The kernel of the used solving algorithm is that solutions have been found for different external influences.

In addition, it is possible to specify a number of articles committed to the research of vibratory procedures in conical shells based on the Timoshenko type theory (Meish & Belova, 2020; Lugovyi & Skosarenko, 2022; Meish et al., 2014; Zeighampour et al., 2015). At the same, linear equations (Meish & Belova, 2020) of the concept of conical shells of Timoshenko type in a orthogonal curved coordinate system are derived, shell theories and Timoshenko hypotheses (Lugovyi and Skosarenko, 2022) are used to develop a method for figuring out the character of oscillations. The stress-strain state of orthotropic ribbed conical shells, problems (Meish et al., 2014) on non-axisymmetric vibrations of a non uniform a conical shell of variable thickness under unstable load using the theory of Timoshenko-type shells. An example of using the shear shell model to acquire the resolving equations of the conical shell vibration is the article (Zeighampour et al., 2015). Lately, numerous research projects have been completed on the study of vibrations of conical shells based on frequency analysis (Xiaohan & Yiling, 2020), a vibrational approach (Yegao et al., 2013), as well as taking into account the influence of rotation (Lam & Hua, 1997; Sarkheil & Mahmoud, 2016; Qinkai & Fulei, 2013).

Thus, based on the above brief overview of the scientific bibliography on the dynamics of conical shells, it can be concluded that at present the study of nonstationary TV of TCS interacting with the surrounding deformable medium is an urgent task. Therefore, this work is dedicated to the derivation of equations of nonstationary TV of a reduced conical shell located in a deformable medium, the creation of an algorithm to determine the shell's VAT across the specified functions field, the analysis of restricting and unique situations of the results achieved, as well as tackling the issue of harmonic vibrations of a truncated conical rod.

2. Formulation of the task

The problem of TV of a TCS with an angle of inclination is considered forming a cone to the axis of symmetry (angle of attack) of the cone, located in an infinite deformable medium (**Fig.1**). The materials of the shell and the medium are assumed to be elastic, homogeneous and isotropic, and the length of the shell is unlimited. The shell is assigned to an orthogonal coordinate system in cylinders (r, θ, z), its beginning is situated on the left end, and the axis is straight the horizontal *Oz* of symmetry of the shell. It is presumed the case $r_1, r_2 -$ radius of the shell are linear functions of the longitudinal coordinate in the form. $r_1 = r_0 + fz,\ \ r_2 = r_0 + h + fz$, where $r_0 = const\ \ h -$is the shell thickness; $f = \tan\varphi$.

In the future, when deriving the vibration equations, probably the TCS, its equations of motion characterize it as a conical three – dimensional entity that adheres rigorously to the mathematical linear theory of elasticity. When addressing axisymmetric non-stationary vibration problems involving circular, cylindrical, and conical structures, the problem of their TV can be taken into account separately from the difficulty of their longitudinal-radial oscillations. In the case of TV of a truncated conical layer, conversing with a deformable elastic medium. Because of the symmetry in the issue relative to the symmetry axis of the conical layer, the components of the stress and deformation tensors are unrelated on the angular coordinate and just the stress component $\sigma_{r\theta}^m, \sigma_{z\theta}^m,\ m = 0{,}1$ and the corresponding deformation components will be nonzero $\varepsilon_{r\theta}^m, \varepsilon_{z\theta}^m, m = 0{,}1$. Only the torsional components $U_\theta^{(m)}$ of the displacement vectors of the conical layer $(m = 0)$ and the medium $(m = 1)$, will be nonzero, that are articulated in using of the components Ψ_m of the vector potential of transverse waves (Khudoynazarov et al., 2022)

$$U_\theta^{(m)} = -\frac{d}{dr}\Psi_m, \quad m = 0{,}1.$$

Hence it follows, that the TV of the truncated conical layer (*m*=0) and the deformable medium surrounding it $(m = 1)$ are described by wave equations with respect to potentials $\Psi_m, m = 0{,}1$

$$\mu(\Delta\Psi_m) = \rho\ddot{\Psi}_m\ ;\ m = \begin{cases} 0, & at\ \ r_1 \le r \le r_2, \\ 1, & at\ \ r_2 \le r < \infty. \end{cases} \tag{1}$$

To create the boundary conditions on the layer's conical surface, formulated introducing an orthogonal coordinate system at an arbitrary point (n, s_1, s_2) on its surface (**Fig. 1**), where n is normal vector; s_1, s_2 − perpendicular to the normal, coordinates in the plane of tangent of the layer attracted to its exterior at the selected point (Khudoynazarov, Kholikov et al., 2022). At the same time, s_1 is directed in the circumferential direction, and s_2 in the longitudinal direction. To formulate boundary conditions at points on a conical surface in an orthogonal coordinate system (n, s_1, s_2), tangential stresses τ_{ns_1} and τ_{ns_2} are expressed in terms of stress components in a cylindrical(r, θ, z) systems. In addition, it is believed (Khudoynazarov & Ismoilov, 2023), that TV of the rod are energized by dynamic outside forces $f_{ns_1}^{(i)}, (i = 1,2)$, acting on the shell's surfaces, that is, in the system (n, s_1, s_2) there are: boundary condition

$$\sigma_{r\theta}^{(0)} - f\sigma_{z\theta}^{(0)} = \sqrt{1+f^2} f_{ns_1}^{(1)}(z,t), \quad at \ r = r_1 \tag{2}$$

dynamic contact condition on the interface of media

$$\sigma_{r\theta}^{(0)} - f\sigma_{z\theta}^{(0)} = \sigma_{r\theta}^{(1)} - f\sigma_{z\theta}^{(1)} + \sqrt{1+f^2}\, f_{ns_1}^{(2)}(z,t), \qquad at \ r = r_2 \tag{3}$$

and kinematic contact condition

$$U_\theta^{[0]}\Big|_{r=r_2} = U_\theta^{[1]}\Big|_{r=r_2} \tag{4}$$

It is assumed that the initial circumstances are zero.

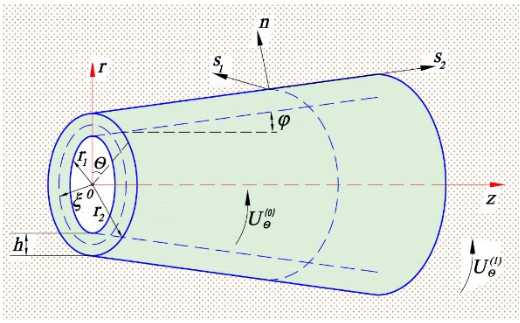

Fig. 1. Conical shell in an elastic medium

3. Vibration equations

In order to resolve the wave Eq. (1), the functions of external influences $f_{ns_1}^{(i)}(z,t), (i = 1,2)$, will be regarded as a member of the class of functions that may be represented in the form of in the boundary and contact criteria (2)-(4).

$$f_{ns_1}^{(i)}(z,t) = \int_0^\infty \left. \begin{matrix} \sin kz \\ -\cos kz \end{matrix} \right\} dk \int_{(l)} \bar{f}_{ns_1}^{(i)}(k,p) e^{pt} dp, \quad (i = 1,2) \tag{5}$$

where (l) - open contour within the plane p, next to the imaginary axis's segment $(-i\omega_0, i\omega_0)$ on the right. Furthermore, the functions $\bar{f}_{r\theta}^{(i)}(k,p)$ are negligible outer the region $k \le k_0$, $|\mathrm{Im} p| \le \omega_0$, which are required in order to obtain the vibration equations (Khudoynazarov, 2003).

According to the illustrations of the roles played by outside factors (5), we represent the potentials Ψ_m and displacements $U_\theta^{(m)}$ as well as Eq. (5)

$$\left[U_\theta^{(m)}, \Psi_m\right] = \int_0^\infty \left. \begin{matrix} \sin kz \\ -\cos kz \end{matrix} \right\} dk \int_{(l)} \left[\bar{U}_\theta^{(m)}, \bar{\Psi}_m^{(0)}\right] e^{pt} dp, \qquad m = 0,1, \tag{6}$$

here through $\bar{U}_\theta^{(m)}(r,k,p)$ and $\bar{\Psi}_\theta^{(m)}(r,k,p)$ denote the images of the potential of displacements $U_\theta^{(m)}(r,z,t)$ and potentials $\Psi_m(r,z,t)$. Substituting representations (6) to equations (3), ordinary differential Bessel equations are obtained, the overall solutions of which, taking into account the roundedness of solutions at infinity, are equal to

$$\bar{\Psi}_0^{(0)} = A_1 I_0(\beta_0 r) + A_2 K_0(\beta_0 r), \quad at \ r_1 \le r \le r_2,$$
$$\bar{\Psi}_1^{(0)} = B K_0(\beta_1 r), \qquad at \ r_2 \le r < \infty. \tag{7}$$

where I_o and K_o −are modified Bessel functions: A_1, A_2 and B −are integration constants;

$$\beta_m^2 = \frac{p^2}{b_m^2} + k^2, \quad b_m^2 = \frac{\mu_m}{\rho_m}, \quad m = 0,1. \tag{8}$$

Let us express the image of the movement of points of the TCS $\bar{U}_\theta^{(0)}(r,k,p)$ through the general solution for $\bar{\Psi}_\theta^{(0)}$ in (7) and highlight its main parts. For this purpose, we substitute solution from Rq. (7) for the shell into the expression for $U_\theta^{(m)}$ at $m = 0$, having previously transformed the displacement according to Eq. (6) and using standard expansions of the modified Bessel functions I_1 and K_1 in powers of the radial coordinate *r* into power series, we obtain

$$\bar{U}_\theta^{(0)} = \frac{1}{r}A_2 - \sum_{n=0}^{\infty} \beta_0^{2n+2}\left[A_1 - A_2\left(ln\frac{\beta r}{2} - \frac{1}{2}[\psi(n+1) + \psi(n+2)]\right)\right]\frac{(r/2)^{2n+1}}{n!\,(n+1)!}, \tag{9}$$

This is ψ(n)- derivative of the Gamma function with logarithmic.

Now, as the main required quantities, we calculate the point displacements of the conical shell's intermediate surface, the radius of which is determined by the formula (Khudoynazarov & Ismoilov, 2023).

$$\xi = \frac{r_1}{2}\left(\chi - \frac{r_1}{r_2}\right), \quad \frac{r_1}{r_2} + 2 \le \chi \le \frac{r_1}{r_2} + 2\frac{r_2}{r_1} \tag{10}$$

In this case, the radius ξ based on the values of the constant χ able to take on the values of the radius of the shell's middle, outer, or surface. Next, we substitute into Eq. (9) $r = \xi$ and confine ourselves to the zero approximation in infinite series. Let's introduce the following new desired functions by the formulas

$$\bar{U}_{\theta,0}^{(0)} = -\frac{1}{2}\beta_0^2\left\{A_1 - A_2\left[ln\frac{\beta_0\xi}{2} - \psi(1) - \frac{1}{2}\right]\right\}, \qquad \bar{U}_{\theta,1}^{(0)} = \frac{1}{\xi}A_2. \tag{11}$$

Substituting Eq. (11) into Eq. (9) for the transformed displacement $\bar{U}_\theta^{(0)}$, yields the following,

$$\bar{U}_\theta^{(0)}(r,z,t) = 2L_r^{(1)}(\beta_0^{2n})\bar{U}_{\theta,0}^{(0)} + \xi\left[\frac{1}{r} + L_r^{(1)}\left(\eta_{1,n}(r)\beta_0^{2n+2}\right)\right]\bar{U}_{\theta,1}^{(0)}, \tag{12}$$

where

$$L_r^{(k)}(\varphi) = \sum_{n=0}^{\infty}\varphi\frac{(r/2)^{2n+k}}{n!\,(n+k)!}; \ (k = 1,2), \qquad \eta_{1,n}(r) = \ln\left(\frac{r}{\xi}\right) + \frac{n}{2n+2} - \sum_{m=1}^{n}\frac{1}{m}. \tag{13}$$

Now let us explain the stress components in terms of the new functions introduced by formulas (11). To do this, we first represent the stresses also in the form Eq. (6), then determining the constant *B* from the contact condition in Eq. (4) and applying Eq. (14) and Eq. (6) to the stress expressions through potential functions $\Psi_m(r)$ and substituting Eq. (7) into the resulting formulas, we will have expressions for the transformed stress components $\bar{\sigma}_{r\theta}^{(m)}$ and $\bar{\sigma}_{z\theta}^{(m)}$, through common solutions. Further, decomposing the modified Bessel functions included in the expressions obtained in this way into power series by degrees of the radial coordinate, as well as excluding constants A_1 and A_2 according to formulas (11), we obtain

$$\bar{\sigma}_{r\theta}^{(0)} = \mu_0\left\{L_r^{(2)}\left[\beta_0^{2n+2}\left(2\bar{U}_{\theta,0}^{(0)} + \xi\eta_{2,n}(r)\beta_0^2\bar{U}_{\theta,1}^{(0)}\right)\right] + \frac{\xi}{2}\left(\beta_0^2 - \frac{4}{r^2}\right)\bar{U}_{\theta,1}^{(0)}\right\},$$
$$\bar{\sigma}_{z\theta}^{(0)} = \mu_0\left\{L_r^{(1)}\left[\beta_0^{2n}\left(2k\bar{U}_{\theta,0}^{(0)} + \xi\eta_{1,n}(r)k\beta_0^2\bar{U}_{\theta,1}^{(0)}\right)\right] + \frac{\xi}{r}k\bar{U}_{\theta,1}^{(0)}\right\},$$
$$\bar{\sigma}_{r\theta}^{(1)} = \mu_1\bar{R}\left\{L_r^{(1)}\left[\beta_0^{2n}\left(2\bar{U}_{\theta,0}^{(0)} + \xi\eta_{1,n}(r)\beta_0^2\bar{U}_{\theta,1}^{(0)}\right)\right] + \frac{\xi}{r}\bar{U}_{\theta,1}^{(0)}\right\},$$
$$\bar{\sigma}_{z\theta}^{(1)} = \mu_1\left\{L_r^{(1)}\left[\beta_0^{2n}\left(2k\bar{U}_{\theta,0}^{(0)} + \xi\eta_{1,n}(r)k\beta_0^2\bar{U}_{\theta,1}^{(0)}\right)\right] + \frac{\xi}{r}k\bar{U}_{\theta,1}^{(0)}\right\} \tag{14}$$

where $\mu_i (i = 0,1)$- shear coefficients of the layer and medium materials, accordingly;

$$\bar{R} = \beta_1\frac{\overline{K_2}(\beta_1 r_2)}{\overline{K_1}(\beta_1 r_2)}; \quad \eta_{2,n}(r) = ln\frac{r}{\xi} + \frac{n^2+n-1}{2(n^2+3n+2)} - \sum_{j=1}^{n}\frac{1}{j}; \tag{15}$$

Next, we introduce functions $U_{\theta,i}^{(0)}(r,z,t),\ (i=0{,}1)$ and operators $\lambda_0^n, n=0{,}1{,}2,\ldots$ according to the following formulas

$$U_{\theta,i}^{(0)}(r,z,t)=\int_0^\infty \left.\begin{matrix}\sin kz\\ -\cos kz\end{matrix}\right\} dk \int_{(l)} \bar{U}_{\theta,i}^{(0)}(r,k,p)e^{pt}dp,\quad (i=0{,}1), \tag{16}$$

$$\lambda_m^n(\zeta)=\int_0^\infty \left.\begin{matrix}\sin kz\\ -\cos kz\end{matrix}\right\} dk \int_{(l)} (\beta_m^{2n}\bar{\zeta})e^{pt}dp\,,\ \ m=0{,}1; n=0{,}1{,}2,\ldots \tag{17}$$

here $\bar{\zeta}$ is one of functions $\bar{U}_{\theta,i}^{(0)}(r,k,p),(i=0{,}1)$. Applying Eqs. (16-17) to Eq. (14) and substituting into the boundary-(2) and contact-(3) conditions, we obtain

$$\begin{gathered}L_{r_1}^{(1)}\left[\lambda_0^n\left(\frac{r_1}{n+2}\lambda_0-r_1 f\frac{\partial}{\partial z}\right)U_{\theta,0}^{(0)}+\xi\lambda_0^{n+1}\left(\frac{r_1}{2(n+2)}\eta_{2,n}(r_1)\lambda_0-\eta_{1,n}(r_1)f\frac{\partial}{\partial z}\right)U_{\theta,1}^{(0)}\right]+\\ \left(\frac{\xi}{2}\left(\lambda_0-\frac{4}{r_1^2}\right)-\frac{\xi}{r_1}\frac{\partial}{\partial z}\right)U_{\theta,1}^{(0)}=\frac{\Delta_0}{\mu_0}f_{ns_1}^{(1)}(z,t),\quad r_1=r_0+fz,\\ L_{r_2}^{(1)}\left[\lambda_0^n\left(\frac{r_2}{n+2}\lambda_0-r_2 f\frac{\partial}{\partial z}\right)U_{\theta,0}^{(0)}+\xi\lambda_0^{n+1}\left(\frac{r_2}{2(n+2)}\eta_{2,n}(r_2)\lambda_0--\eta_{1,n}(r_2)f\frac{\partial}{\partial z}\right)U_{\theta,1}^{(0)}\right]+\\ \left(\frac{\xi}{2}\left(\lambda_0-\frac{4}{r_2^2}\right)-\frac{\xi}{r_2}\frac{\partial}{\partial z}\right)U_{\theta,1}^{(0)}=\\ =\frac{\mu_1}{\mu_0}\left(R-f\frac{\partial}{\partial z}\right)\left\{L_{r_2}^{(1)}\left[2\lambda_0^n U_{\theta,0}^{(0)}+\xi\lambda_0^{n+1}\eta_{1,n}(r_2)U_{\theta,1}^{(0)}\right]+\frac{\xi}{r_2}U_{\theta,1}^{(0)}\right\}++\frac{\Delta}{\mu_0}f_{ns_1}^{(2)}(z,t),\qquad r_2=\\ r_0+h+fz,\Delta_0=\sqrt{1+f^2}\,;R\approx\frac{2}{r_2}+(1-\gamma)r_2\lambda_1;\end{gathered} \tag{18}$$

here operator R - is the original image operator $\bar{R}$. As follows from the form (18) of operator R, it represents a dynamic reaction of an elastic medium to vibrations of a TCS.

Based on Eq. (16) and Eq. (17), from Eq. (12) we obtain a formula expressing the displacement $\bar{U}_\theta^{(0)}(r,z,t)$ through the functions $U_{\theta,0}^{(0)}$ and $U_{\theta,1}^{(0)}$

$$U_\theta^{(0)}(r,z,t)=2L_r^{(1)}(\lambda_0^n)U_{\theta,0}^{(0)}+\xi\left[\frac{1}{r}+L_r^{(1)}\left(\eta_{1,n}(r)\lambda_0^{n+1}\right)\right]U_{\theta,1}^{(0)}, \tag{19}$$

for $r=\xi$ and $n=0$ it follows from Eq. (19) that

$$U_\theta^{(0)}(\xi)=U_{\theta,1}^{(0)}+\xi U_{\theta,0}^{(0)},$$

which shows that the introduced new functions are the primary components of the conical layer's intermediate surface's torsional displacement and, in this case, the function $U_{\theta,1}^{(0)}$ has the dimension of displacement, and the function $U_{\theta,0}^{(0)}$ has the dimension of deformation.

From representations (8) for $\beta_m^2, m=0{,}1,$ it is easy to see that the previously introduced operators λ_m^n variables (z,t) get the following form (Filippov and Kudajnazarov, 1998)

$$\lambda_m^n(\ldots)=\left[\frac{1}{b_m^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right]^n,b_m=\sqrt{\frac{\mu_m}{\rho_m}},\ \ m=0{,}1;\ n=0{,}1{,}2,\ldots \tag{20}$$

It follows that the operators $\lambda_m(\ldots),\lambda_m^2(\ldots),\ldots$. are differential operators of the second, fourth, and so on orders, and the operators $\lambda_m^0(\ldots)=1$. Equations (18) in accordance with expression (20) for operators λ_m^n there is a system of differential equations for the main parts $U_{\theta,0}^{(0)}$ and $U_{\theta,1}^{(0)}$ torsional displacement of a TCS. It represents the general equations of nonstationary TV of a truncated conical elastic shell, which are dependent upon the operators λ_m^n and the primary components of the torsional displacement of the layer's intermediate surface locations.

4. Particular cases of equations of vibration of a TCS

The obtained general Eq. (18) of TV of a TCS allow some cases of a particular form, namely. From them, it is possible to obtain the equations of TV of a truncated conical thin-walled shell, a truncated conical rod and a circular cylindrical layer with the corresponding particular types (round rod and circular cylindrical thin-walled shell). Listed below are some of the indicated particular types of Eq. (18) (Yalgashev et al., 2022).

4.1 Estimate TCS equations vibration.

Thus, it follows that restricting the number of terms in the system of Eq. (18), i.e. constrain them to $(n = 0)$, $(n = 1)$, $(n = 2)$, etc. approximations. Assuming that the conditions for the convergence of infinite series consisting of operators (13) -$L_{r_i}^{(m)}, (i, m = 1,2)$, i.e. the conditions (Khudoynazarov, 2003), are met regarding the range of application of the formulas "truncated" in this way, let us analyse their zero approximation. Assuming $(n = 0)$ in system (18), we will have approximate equations for nonstationary TV of a TCS located in an elastic medium

$$\begin{gathered} r_1\left[\frac{r_1}{4}\lambda_0 - f\frac{\partial}{\partial z}\right]U_{\theta,0}^{(0)} + \xi\left\{\frac{1}{2}\lambda_0 - \frac{2}{r_1^2} - \frac{f}{r_1}\frac{\partial}{\partial z} + \frac{r_1}{2}\lambda_0\left(\eta_{2,0}(r_1)\frac{r_1}{4}\lambda_0 - \right.\right. \\ \left.\left. - f\eta_{1,0}(r_1)\frac{\partial}{\partial z}\right)\right\}U_{\theta,1}^{(0)} = [1+f^2]\mu_0{}^{-1}[f_{ns_1}^{(i)}(z,t)]; \\ r_2\left[\frac{r_2}{4}\lambda_0 - \frac{\mu_1}{\mu_0}\left(R - f\frac{\partial}{\partial z}\right) - f\frac{\partial}{\partial z}\right]U_{\theta,0}^{(0)} + \xi\left\{\frac{1}{2}\lambda_0 - \frac{2}{r_2^2} - \frac{1}{r_2}\frac{\mu_1}{\mu_0}\left(R - f\frac{\partial}{\partial z}\right) - -\frac{f}{r_2}\frac{\partial}{\partial z} + \right. \\ \left.\frac{r_2}{2}\lambda_0\left(\eta_{2,0}(r_2)\frac{r_2}{4}\lambda_0 - f\eta_{1,0}(r_2)\frac{\partial}{\partial z} - \frac{\mu_1}{\mu_0}\left(R\eta_{1,0}(r_2) - f\frac{\partial}{\partial z}\right)\right)\right\}U_{\theta,1}^{(0)} == [1 + \\ f^2]\mu_0{}^{-1}[f_{ns_1}^{(i)}(z,t)]; \end{gathered} \tag{21}$$

where

$$\eta_{1,0}(r_1) = ln\frac{r_1}{\xi};\ \eta_{2,0}(r_2) = ln\frac{r_2}{\xi} - \frac{1}{4};\ (i = 1,2). \tag{22}$$

The derived equations are approximations of the TV of an elastic shell with a truncated conical shape located in an elastic medium.

4.1.1 Truncated conical thin-walled shell equation

Let us assume that $r_2 = r_1 + \varepsilon$, where $\varepsilon > 0$ is a small value that satisfies the condition of thinness of the shell walls. Then, in expressions (22), we can assume that $ln(r_k/\xi) = 0, (k = 1,2)$, taking into account which Eq. (21) are greatly simplified. The resulting equations will be the equations of TV of a truncated conical thin-walled shell interacting with an elastic medium. At $R = 0$, they transform into the well-known vibration equations obtained in (Khudoynazarov et al., 2022).

4.2 TV of a circular cylindrical shell equation

If $f = 0$, i.e. the angle of attack is zero, then the conical shell passes within an elastic medium located in a cylindrical shell, and the vibration Eq. (21) take the form

$$\begin{gathered} L_{r_2}^{(1)}\left[\frac{r_2}{n+2}\lambda_0^{n+1}U_{\theta,0}^{(0)} + \frac{r_2\xi}{n+2}\eta_{2,n}(r_i)\lambda_0^{n+2}U_{\theta,1}^{(0)}\right] + \left(\frac{\xi}{2}\left(\lambda_0 - \frac{4}{r_2^2}\right) + + \frac{\xi}{r_2}\frac{\partial}{\partial z}\right)U_{\theta,1}^{(0)} = \\ \frac{[1+f^2]}{\mu_0}f_{ns_1}^{(i)}(z,t) + \frac{\mu_1}{\mu_0}R_i\left\{L_{r_2}^{(1)}[2\lambda_0^n U_{\theta,0}^{(0)} + +\xi\lambda_0^{n+1}\eta_{1,n}(r_2)U_{\theta,1}^{(0)}] + \frac{\xi}{r_2}U_{\theta,1}^{(0)}\right\}, R_i = \\ \begin{cases} 0, & i = 1, \\ \frac{2}{r_2} + (1-\gamma)r_2\lambda_1; & i = 2. \end{cases} \end{gathered} \tag{23}$$

As a result, the system exactly coincides with the vibration equations in (Khudoynazarov, 2003). Note that Eq. (23) allow limiting cases of a thin-walled cylindrical shell and a round rod (Khudoynazarov, Kholikov et al., 2022).

4.3 Equations of a truncated conical rod

Let $r_1 = 0$, then in accordance with formula (10) the equality will take place $\xi = 0$. Then the conical shell under consideration will be transformed into a conical rod of radius $r_2' = h + fz$. In this case, the radius of the left end of the rod at point $z = 0$ will be equal to h, i.e. $r_2' = h$. Substituting $r_1 = 0$ and $\xi = 0$ into (18), we acquire the equations of unsteady TV of a truncated cone rod located in an elastic medium

$$\begin{gathered} L_{r_2}^{(1)}\left[\lambda_0^n\left(\frac{r_2'}{n+2}\lambda_0 - r_2' f\frac{\partial}{\partial z}U_{\theta,0}^{(0)}\right)\right] = \frac{\mu_1}{\mu_0}\left(R - f\frac{\partial}{\partial z}\right)L_{r_2}^{(1)}[2\lambda_0^n U_{\theta,0}^{(0)}] + \frac{\Delta_0}{\mu_0}f_{ns_1}^{(2)}(z,t), \\ r_2' = h + fz; \end{gathered} \tag{24}$$

4.4 Equation of S.P. Beridze

Here, in a particular case, $n = 0$ in infinite series of the operator $L_{r_2}^{(1)}$, we obtain an approximate equation that is more general than the well-known equation of a conical rod proposed by S.P. Beridze.

5. Harmonic TV of a truncated conical rod located in an elastic medium

Explore the harmonic TV problem of a conical rod that has been truncated located in an elastic medium. The decisive equation of the problem is Eq. (24) in the zero approximation. To solve the problem of harmonic vibrations, external influences should be considered absent, which is equivalent to the equality of the right parts of the obtained equations to zero. In addition, if we limit ourselves in the infinite series of operator $L_{r_2}^{(1)}$ to the zero approximation ($n = 0$), we get

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} - \frac{8}{(h_0 + fz)^2}\frac{\mu_1}{\mu_0} - 4(1-\gamma)\frac{\mu_1}{\mu_0}\frac{1}{b_1^2}\frac{\partial^2}{\partial t^2} + \right.$$
$$\left. +4(1-\gamma)\frac{\mu_1}{\mu_0}\frac{\partial^2}{\partial z^2} - \frac{4}{h_0 + fz} f\frac{\partial}{\partial z}\left(1 - \frac{\mu_1}{\mu_0}\right)\right] U_{\theta,0}^{(0)} = 0 \tag{25}$$

The transition in Eq. (25) to dimensionless variables in accordance with the formulas

$$z = h_0 z^*;\ t = (h_0/b_0)t^*; b_1 = b_0 b_1^*;\ U_{\theta,0}^{(0)} = U$$

we get

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} - \frac{8}{(1 + fz)^2}\frac{\mu_1}{\mu_0} - 4(1-\gamma)\frac{\mu_1}{\mu_0}\frac{1}{b_1^2}\frac{\partial^2}{\partial t^2} + \right.$$
$$\left. +4(1-\gamma)\frac{\mu_1}{\mu_0}\frac{\partial^2}{\partial z^2} - \frac{4}{1 + fz} f\frac{\partial}{\partial z}\left(1 - \frac{\mu_1}{\mu_0}\right)\right] U_{\theta,0}^{(0)} = 0 \tag{26}$$

We will look for solutions for the system (26) in the form of

$$U = Ae^{i(\omega t + kz)} \tag{27}$$

where A -is constants; $k -$ wave number; $\omega -$ circular frequency of vibrations; $i -$ imaginary unit. Substituting (27) into Eq. (25), we obtain the frequency equation.

$$\left(-1 + 4(1-\gamma)\frac{\mu_1}{\mu_0}\frac{1}{b_1^2}\right)\omega^2 + \left(1 - 4(1-\gamma)\frac{\mu_1}{\mu_0}\right)k^2 - \frac{4}{1 + fz} f\left(1 - \frac{\mu_1}{\mu_0}\right)ki - \frac{8}{(1 + fz)^2}\frac{\mu_1}{\mu_0} = 0 \tag{28}$$

Eq. (18) solved with Maple 17 application software package for the values that follow of geometrical, physical and mechanical parameters of the materials of a truncated conical rod and its environment:

1) angle of attack values -0.5^0, 1^0, 1.5^0 , 2^0;
2) shell materials: a) steel (E=2·10^8 kPa; ν=0.25; ρ=7850 kg/m^3), b) aluminum (E=7·10^7 kPa; ν=0.35; ρ=2750 kg/m^3), c) copper (E=1·10^8 kPa; ν=0.31; ρ=8940 kg/m^3);
3) environmental materials: a) sand (E=4·10^7 Pa; ν=0.31; ρ=2000 kg/m^3), b) dense clay (E=7.6·10^9 Pa; ν=0.32; ρ=2120 kg/m^3), c) loam (E=1.8·10^9 Pa; ν=0.35; ρ=1650 kg/m^3).

6. Numerical results and discussions.

The quantitation results obtained are presented in **Figs. 2-6** in the form of graphs of the dependencies of the circular frequency ω of a truncated conical rod on the wave number k. **Fig. 2** shows the dependences of the dimensionless vibration frequency ω on the wave number k for various materials of a truncated conical rod at an angle of attack of 0.5 degrees. At the same time, the physical and mechanical characteristics of Aluminium, Copper and Steel listed above are used as materials for the shell. For calculations, it is assumed that there is no interacting medium ($\mu_1 = 0$ in Eq. (24)). It can be seen that the higher the value of the elastic modulus of a material (steel), the lower the frequency of vibrations of this material, and vice versa, the shell, the material of which has a lower value, has a higher frequency of vibrations. For example, with a wave number value of $k = 2$, the difference between the vibration frequencies of aluminium and steel shells is 27%. With the growth of the wave number, this difference increases and reaches, for example, at $k = 4$, 32%.

For comparative analysis, **Fig. 3** shows the vibration frequencies of a round cylindrical rod, when the angle of attack is zero ($f = 0$ in Eq. (24)), made of the same materials and also in the absence of an interacting medium ($\mu_1 = 0$). As can be seen from the above graphs, which are straight lines coming from the origin, in the absence of an interacting medium and the angle of attack is zero, we get a classic result (a directly proportional dependence of the vibration frequency of a circular elastic rod on the wavenumber). This result confirms the fact that the obtained equations of vibrations of a circular conical elastic rod in Eq. (24), in the unique instance of a circular elastic rod where there is no external environment, correctly describe the harmonic process of TV. In addition, it follows from the comparisons of **Fig. 2** and **Fig. 3** that the presence of a taper ($f \neq 0$ in Eq. (24)) leads to a violation of the direct proportionality between the vibration frequency and the wave number in the case of a conical rod.

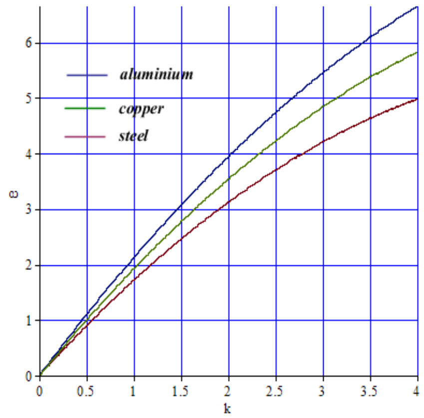

Fig. 2. Dependences of the dimensionless vibration frequency on the wave number for various materials of a truncated conical rod at the angle of attack 0.5 degree.

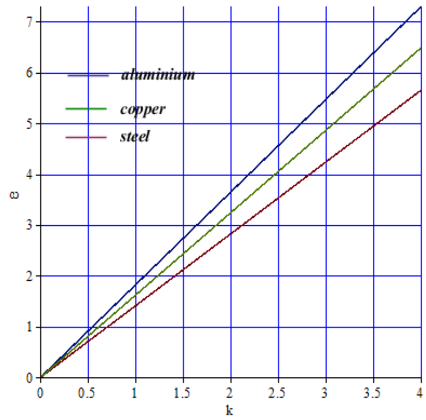

Fig. 3. Dependence of the dimensionless vibration frequency on the wave number for various materials of a round cylindrical rod (the angle of attack is zero).

Fig. 4 shows the dependences of the dimensionless vibration frequency ω on the wave number k for various materials of a truncated conical rod located in loam at an angle of attack of 0.5 degrees. From comparing the numerical results obtained with the results in **Fig. 2**, it follows that the presence of an external interacting medium leads to an even greater violation of the above-mentioned property of proportionality between the vibration frequency and the wavenumber. In this case, even the order of the frequency curves for different materials is violated. For example, in this case, the curves of the dependence of the oscillation frequency of an aluminium truncated conical rod in the entire range of values of the wave number k became smaller than the frequencies of the copper and steel shells. In addition, the nature of the function of the wavenumber also changes. If, in the absence of an external elastic medium, the concavity of the frequency curves is directed downward, then in the case of an external elastic medium, the concavity of the frequency curves is directed upward.

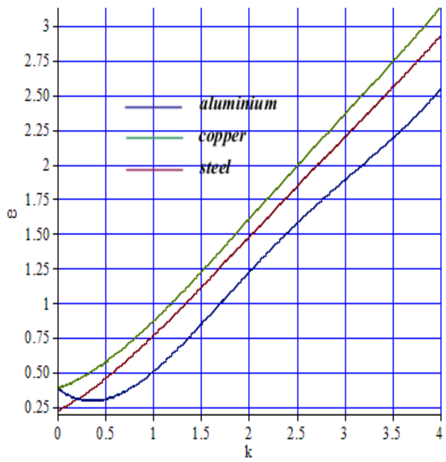

Fig. 4. The dependences of the dimensionless vibration frequency on the wavenumber for various materials of a truncated conical rod located in loam at an angle of attack of 0.5 degrees.

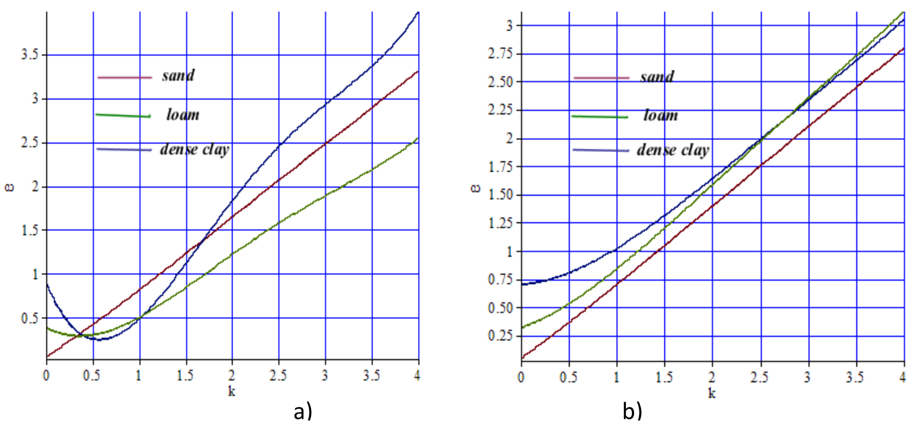

Fig. 5. materials: a) aluminium b) copper, at an angle of attack of 0.5 degrees. The dependences of the dimensionless vibration frequency ω on the wave number k located in an elastic medium (sand, loam and dense clay) of a truncated conical rod for its various.

Fig. 5a,b shows the dependences of dimensionless vibration frequencies ω on the wave number k located in an elastic medium (sand, loam and dense clay) of a truncated conical rod for its various materials: a) aluminium b) copper, at an angle of attack of 0.5 degrees. It follows from the above graphs that in the case of aluminium (**Fig. 5a**) and in the case of copper (**Fig. 4b**), the graphs of the dependences of the dimensionless vibration frequency ω on the wave number k located in the sand medium (red lines) are straight lines. This effect is explained by the fact that the modulus of elasticity of sand (E=4·10^7 Pa) is several thousand times less than the modulus of elasticity of the rod material: aluminium (E=7·10^{10}Pa) or copper (E=1.0·10^{11} Pa).

The graphs show that the higher the density of the surrounding elastic medium, the higher the vibration frequencies of the conical rod. For instance, in the case of aluminium with a wave number value of $k = 4$, the frequency value of a rod located in dense clay is 54% higher than a rod located in loam (**Fig. 5a**). In the case of a copper rod, the oscillation frequency curves of a conical rod are arranged from bottom to top when the densities of the media surrounding the rod are located from small to large (**Fig. 5a, b**). At the same time, it should be noted that the value of the frequency of TV everywhere, in the region of modifications in the wave number, is different from zero, even if the wave number is 0, i.e. at $k = 0$, (**Fig. 4a, b**). Only when the external environment is sand (red lines in **Fig. 5a, b**) these values are close to zero. In addition, it can be seen that the "softer" the material (aluminium **Fig.4a**), the higher the vibration frequency of the conical rod made of this material. On the contrary, if the material is "tougher", then the vibration frequencies of a conical rod made of such a material are much lower compared to a soft material

Fig. 6a,b demonstrates the curves of the dependence of the dimensionless vibration frequency ω on the wave number k of a truncated conical rod for various materials: a) aluminium b) copper, at different angles of attack and in the absence of an external environment. Of these it follows that regardless of the material (**Fig.5a, b**) of the conical rod, without taking into account the external interacting medium, with increasing values of the angle of attack, the torsional vibration frequencies of the circular conical elastic rod increase. For example, in the case of an aluminium rod (**Fig.5a**) with a wavenumber value of $k = 3$, the frequency value for $f = 2^0$ differs from the value for $f = 0.5^0$ by 13.4%, and in the case of a copper rod by 15%.

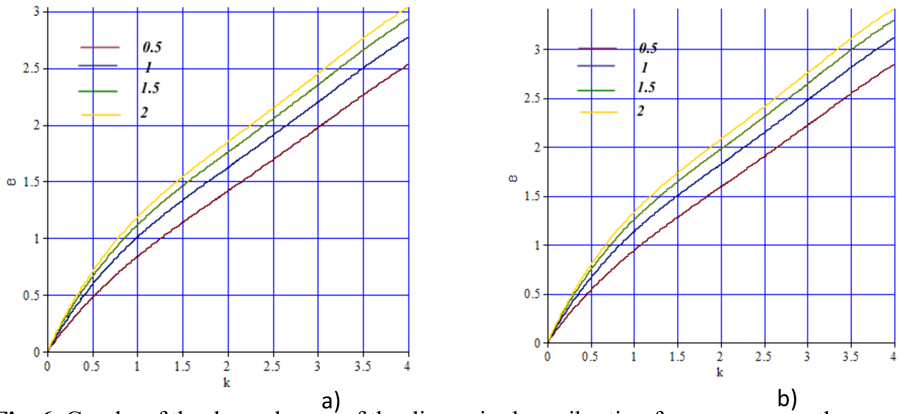

Fig. 6. Graphs of the dependences of the dimensionless vibration frequency ω on the wave number of a truncated conical rod for various materials: a) aluminium b) copper, at different angles of attack and in the absence of an external environment.

7. Conclusion

- TV problem of a TCS interacting with an external elastic medium is formulated and solved;
- For a "certain" intermediate shell surface for main components of torsional displacement, the basic and approximated equations for TV of a TCS are obtained;
- derived system of differential equations of TV of a TCS, in special case, passes into thin-walled shell and rod vibration differential equation;
- based on the derived rod equations, the problem of harmonic TV of a truncated conical elastic rod interacting with an external elastic medium is solved;
- the obtained numerical results make it possible to conclude that the applied theory of the specified rods in a deformable medium characterizes the vibratory motion at different angles of attack quiet well.

References

Alijani, F., & Amabili, M. (2014). Non-linear vibrations of shells: A literature review from 2003 to 2013. *International Journal of Non-Linear Mechanics*, *58*, 233–257.

Aris, H., & Ahmadi, H. (2020). Nonlinear vibration analysis of FGM truncated conical shells subjected to harmonic excitation in thermal environment. *Mechanics Research Communications. 104,* 103499.

Bakhtiari, M., & Lakis, Y, K. (2019). Nonlinear Vibration of Truncated Conical Shells: Donnell, Sanders and Nemeth Theories. *International Journal of Nonlinear Sciences and Numerical Simulation, 21*(1). https://doi:10.1515/ijnsns-2018-0377.

Beridze, S. P. (1999). Free torsional vibrations of a conical rod. *Vestnik OGU, 3,* 104 –107.

Filippov, I. G., & Kudajnazarov, K. (1998). Boundary value problems of longitudinal vibrations of the circular cylindrical shells. *Gangue Jinzhou/Industrial Construction, 28*(12)*,* 34–40.

Khalmuradov, R, I., Khudoynazarov, Kh., & Nishanov, U, A. (2022). Elastic-plastic deformation of a round plate reinforced with stiffeners. *Magazine of Civil Engineering. 116*(8). https://doi: 10.34910/MCE.116.13.

Khudoynazarov, Kh. (2003). *Nonstationary interaction of cylindrical shells and rods with a deformable medium*. Tashkent, publishing house of med.

Khudoynazarov, Kh. (2024). Longitudinal-radial vibrations of a viscoelastic cylindrical three-layer structure. *Mechanical engineering*. https://doi.org/10.22190/FUME231219010K.

Khudoynazarov, Kh., & Yalgashev, B. (2021). Longitudinal vibrations of a cylindrical shell filled with a viscous compressible liquid. *E3S Web of Conferences 264,02017*. https://doi.org/10.1051/e3sconf/202126402017.

Khudoynazarov, Kh., Kholikov, D., & Abdurazakov, J. (2022). Torsional vibrations of a conical elastic shell. *AIP Conference Proceedings, 2637,* 030024. https://doi.org/10.1063/5.0118846.

Kuhudoynazarov, Kh., & Ismoilov, Kh. (2023). A mathematical model of torsional vibrations of a truncated conical layer interacting with a deformable medium. *Fundamental and applied problems of engineering and technology*, *6*(362)*,* 160–172.

Kushnarenko, V, M., & Beridze, S, P. (2000). Free longitudinal vibrations of a conical rod, *Vestnik, 3*, 83–86.

Lam, K., & Hua, L. (1997). Vibration analysis of a rotating truncated circular conical shell. *International Journal of Solids and Structures, 34*(17), 2183–2197.

Lugovyi, P, Z., & Skosarenko, Y, V. (2022). Vibrations of a Layered Conical Shell under Nonstationary Load. *International Applied Mechanics, 58,* 280–288.

Marco, A., & Prabakaran, B. (2020). Nonlinear forced vibrations of layered composite conical shells using an improved theory of shear deformation. *Composite structures, 249,* 112522. https://doi:10.1016 / j.compstruct.2020.112522.

Meish, V, F., Galagan, O, G., & Mel'nik, M, N. (2014). Nonaxisymmetric vibrations of conical shells of variable thickness under a nonstationary load. *International Applied Mechanics, 50*(3), 295–302

Meish, V, F., Meish, Y, A., & Belova, M, A. (2020). Nostationary dynamics of elliptic isotropic conical shells under distributed loads. *International Applied Mechanics, 56*(4), 424–431.

Qinkai, H., & Fulei, C. (2013). Effect of rotation on frequency characteristics of a truncated circular conical shell. *Archive of Applied Mechanics, 83,* 17890-1800. https://doi.org/10.1007/s00419-013-0778-x.

Sarkheil, S., & Mahmoud, S, F. (2016). Free vibrational characteristics of rotating joined cylindrical-conical shells. *School of Mechanical Engineering. Sharif University of Technology, 107*, 657–670.

Shekari, A., Faramarz, A, G., & Malekzadehfard, K. (2017). Free Damped Vibration of Rotating Truncated Conical Sandwich Shells Using an Improved High-Order Theory. *Latin. American journal of Solids and Structures. Rio de Janeiro*, *14,* 12. https://doi.org/10.1590/1679-78253977.

Sofiyev, A. H. (2012). The non-linear vibration of FGM truncated conical shells. *Composite Structures 94(2012).* 2237–2245.

Sofiyev, A. H. (2014). Large-amplitude vibration of non-homogeneous orthotropic composite truncated conical shell. *In: Composites Part B: Engineering 61*, 365–374.

Xiaohan, B., & Yiling, L. (2020). Natural frequency analysis and simulation test of conical shell under support condition. *Journal of Physics Conference Series, 1550*(3), 032002. https://doi:10.1088/1742-6596/1550/3/032002.

Yalgashev, B., Ismoilov, E, A., & Khudoyberdiyev, Z. (2022). Torsional Vibrations of Layered Cylindrical Viscoelastic Shells and Rods. *AIP Conference Proceedings, 2637*, 030023. https://doi.org/10.1063/5.0118588. Published Online: 20 October 2022.

Yegao, Q., Yong, Ch., Xinhua, L., Hongxing, H., & Guang, M. (2013). A modified variation approach for vibration analysis of ring-stiffened conical-cylindrical shell combinations. *European Journal of Mechanics A/Solids*, *37*, 200–215.

Zeighampour, H., Yaghoub, T, B., & Fahimeh, M. (2015). A shear deformable conical shell formulation in the framework of couple stress theory. *Acta Mechanica, 226*, 2607–2629. https://doi 10.1007/s00707-015-1318-2.