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Assortment and promotion optimization in a retail chain

Hamed Karimi[a*]

[a]*School of Industrial Engineering, College of Engineering, University of Tehran, Tehran, Iran*

C H R O N I C L E

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A B S T R A C T

An examination of two areas of promotion and assortment planning in an environment is attempted in this paper. Sales promotion is a marketing strategy used by retailers to increase sales and profits by retaining customers and preventing them from switching to their competitors. Various products are available on the market that can substitute each other, so the best product assortment must be determined as well. In order to model the above subject, a nonlinear integer programming problem is proposed. Model solution involves rephrasing the problem as mixed integer linear programming. Small- and medium-sized problems can therefore be solved using MIP solver software. Firefly algorithms are designed to solve large-scale problems. According to the numerical results, determining the best product assortment for stores must also be done simultaneously with finding the optimal promotion. As a matter of fact, the promotion of the products significantly affects the assortment scenarios for the stores. Consequently, the selection of the promotional discount may result in large profit losses if the assortment planning is not taken into consideration. In order to assess the importance and sensitivity of the model parameters, a sensitivity analysis is conducted. The sensitivity analysis demonstrates that the model is able to respond to changes in market demand and competition, and provides an effective tool for chain stores to optimize their promotion and assortment strategies. To further validate the effectiveness of the model, a case study is conducted in Tehran, Iran. The results of the case study demonstrate the ability of the model to effectively optimize promotion and assortment strategies in real-world settings. Overall, the proposed model provides a valuable tool for chain stores to optimize their promotion and assortment strategies, and improve their market competitiveness.

1. Introduction

Competition is recognized as an essential component of every supply chain today, and if any actor in the chain ignores it, their supply chain will be at risk of failing. The choices made in a chain are one of the elements that greatly affect a supply chain's ability to compete. The decisions made by companies in terms of tactical and operational matters are shaped by market shifts and competitions, such as assortment choices, pricing, and promotion plans (Shankar et al., 2013). This paper explores two important factors influencing chain stores' competition: assortment planning and promotion optimization.

The selection of products that maximize profitability is a major issue in revenue management and retail operations (Gallego & Topaloglu, 2019). Assortment planning involves choosing a set of products for clienteles in order to maximize the profit gained by those customers when they purchase the products. Assortment planning involves using models that specify Behavior of substitution and Demand resulting from it based on a collection of products (Désir et al., 2020). Numerous studies have been conducted on assortment optimization that take into account customer choice behavior. It is clear that assortment has an impact on costs since it drives inventory decisions. Inventory costs are raised by poorly designed, inefficient assortments, which also eat up valuable shelf space.

* Corresponding author.
E-mail address hamedkarimi7676@ut.ac.ir (H. Karimi)

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On the other hand, there are several ways retailers can increase profits and sales. One of the most effective marketing methods is sales promotion. The main promotional technique used by retailers is temporarily lowering product prices. In addition to increasing sales, promotions improve customer loyalty and cash flow. Moreover, promotions can directly affect profitability in low-profit industries. Retail promotions can pose a challenge since the more a product is promoted, the greater its likelihood of attracting customers and simultaneously the lower its profit margin. When determining the optimal promotional discount, several factors should be taken into account. Despite the intricacy involved, many managers continue to rely on manual methods for planning promotions in their day-to-day operations (Cohen et al., 2021).

To our knowledge, no research has been conducted on optimizing sales promotions and assortment planning at the same time. Retail sales promotion studies in the past, whether descriptive or mathematical research, emphasized pricing concepts without considering assortment decisions, which have a major impact on promotion activities.

Given the nonlinearity and NP-hardness of our problem formulation, we employ a linearization technique to transform the model into a linear form, enabling faster optimization. By utilizing our model, managers gain the capability to analyze various scenarios and make informed decisions by leveraging the generated outcomes.

The rest of the paper is arranged as follows. We review related literature in Section 2. Section 3 outlines the issue. In Section 4, we introduce both our proposed heuristic approach and the exact solution method. Section 5 presents several computational examples to prove the model and proposed methods are effective. Section 6 provides managerial insight and practical implications of the proposed model for chain stores and Section 7 provides a comprehensive summary of our findings and offers key conclusions and recommendations for future research directions.

2. Literature Review

We commence by separately reviewing the literature related to promotion and assortment planning, subsequently highlighting the research gaps.

2.1. Sales promotion

In the field of marketing, Sales promotion has been the subject of extensive study and investigation over the course of several years. The majority of marketing research focuses on analyzing and estimating dynamic sales models, aiming to provide company management with a comprehensive understanding of the business. Retail sales promotion is empirically examined in several studies (e.g., Felgate & Fearne, 2015). Generally, these works are of a descriptive nature. Numerous studies have explored the impact of retail sales promotions on customer loyalty and purchase decisions (e.g., Amini et al., 2012; Hanaysha, 2018). According to Mendez et al. (2015), With the passage of time, sales promotions foster heightened customer loyalty and bolster the reputation of the brand. A price elasticity approach was proposed by Greenstein-Messica & Rokach (2020) to predict prices in e-commerce retail stores. It was found that retailers and suppliers can increase the efficiency of sales promotions through collaboration by Breiter and Huchzermeier (2015). In Agu's study (2021), customer's willingness to participate in a sales promotion campaign was influenced by perceived transparency. Using online grocery delivery services, Joshi and Bhatt (2021) examine how promotions influence Intentions to purchase groceries, in addition to examining the indirect and direct effects of various mediating factors on the decision to purchase groceries.

In the realm of operations research, there has been a recent focus on optimizing promotional strategies. Cohen et al. (2017) conducted a study to explore the most effective approach for promoting a specific item within an operations research framework.

Hamdani (2022) examined the demonstration of the positive correlation between the level of discounts, customer retention, and brand reputation. Accordingly, increasing discounts significantly contributes to brand reputation enhancement while simultaneously cultivating loyal customers. Ilyas et al. (2022), conducted a study aimed at determining optimal pricing and Promotion to enhance and facilitate customer satisfaction in the context of support services. In their research, they employed regression analysis to dissect and analyze customer behavior. Theoretically, a direct relationship exists between independent variables consisting of prices, advertising, and support services against dependent variables of customer satisfaction.

Nouri-Harzvili and Hosseini-Motlagh (2023) delved into a study on dynamic pricing in an online store and examined the relationship between discounts and inventory levels. In their research, they focus on estimating the optimal discount level while accounting for the impact of inventory levels in the online store. This study demonstrates that the model presented can assist online retailers in dynamically adjusting discounts and selecting optimal discount offers. Mohammadi-Pour et al. (2023) have explored a novel model for optimizing sales promotion in competitive markets and investigated how competition influences business performance and sales promotion planning in retail supply chains. In cases where retail market competition exists, offering similar products with varying discounts, the model proposed in this study aids in determining the optimal advertising discounts for different products. They utilized a non-linear integer programming problem in their study for modeling. To solve this model, they converted it into a mixed-integer linear programming problem. The research findings underscore the importance of considering various competitors in promotion planning and optimization, as neglecting them can result in profit loss.

2.2. *Assortment Planning*

The “assortment problem” was probably first discussed in literature on assortment planning in the 1950s by Sadowski (1959). The model of consumer demand, the pattern of product substitution, decisions regarding inventory levels and considerations of assortment capacity are just a few of the aspects that have been taken into account in each of the related studies.

In the study conducted by Bernstein et al. (2015), prior research is classified based on the model of customer preferences employed, including multinomial logit models (MNLs) which have implications for other facets of the problem. In MNL, the utility associated with each product is decomposed into deterministic and random components for each customer visiting a store. (See, for example, Besbes & Sauré, 2016). Several choice models have been employed to better formulate the behavior of product replacement due to some shortcomings in the MNL model. Examples include nested logit models and multinomial logit models (See, for example, Sen et al., 2018). The demand for each product in an exogenous demand model is determined in advance for the entire assortment, irrespective of the assortment selected (Smith & Agrawal, 2000, for example). Consumer preferences have been depicted in recent work using ranking-based models, so that every customer provides a product ranking they like best (See, for example, Goyal et al., 2016). To gain a comprehensive understanding of these models, Train's (2009) work provides an overview. Subsequently, researchers expanded upon these models by incorporating additional variables. For instance, Kök and Fisher (2007) took into account shelf space limitations, while Yücel et al. (2009) considered both supplier choice and shelf space constraints in their analysis.

Chong et al. (2001) introduced measures that describe the product portfolio at the brand level and explain how customer preferences are extracted for different sets of products and services. Mahajan and van Ryzin (2001) developed a simple random route optimization model for product portfolio planning, considering both dynamic and static substitution approaches. Substitution was based on the principles of maximizing desirability using the MNL model. Agrawal et al. (2002) presented a model for capacity, inventory, and transportation management for a collection of products and services. Random demand, fluctuating over time, was considered. Gaur and Honhon (2005) solved the problem of single-time product portfolio planning and inventory management by considering location selection. Decisions related to diversity, product location, and inventory were determined under static substitution, and then a similar model was developed using the boundary created in the static model for dynamic substitution. Cachon and Kök (2007) studied the problem of product portfolio planning with multiple product groups and basket-buying customers. They developed a theoretical game model in which retailers determined the price level and diversity in each group, and customers made their own choice among different stores and had the possibility of not buying, based on the desirability of the products. Kök and Fisher (2007) developed a method for forecasting demand and parameters and provided an innovative optimization method for solving problems. Li (2007) presented the problem of product portfolio planning using the MNL demand model, which measures continuous or interval traffic within the store. Goyal et al. (2009) showed that even the simple issue of product portfolio planning is NP-Hard. Upon reviewing the existing literature, we have observed that a significant number of past studies have focused on determining sets or subsets of products within a specific time frame. Several articles have centered their attention on selecting a single assortment of products for all retailers. However, it is highly likely that different retailers may opt for different product assortments (Singh & Kapoor, 2013).

Bernstein et al. (2019) considered an online retailer that faces customers with diverse and unspecified preferences. Customers are described by a diverse set of demographic and exchangeable features. By leveraging available profile information and analyzing customer data, the retailer has the ability to customize the assortment of products offered to customers and make estimations about their preferences.

Zhang et al. (2020) considered the capacitated and uncapped product set problem and aimed to identify a range of products that would generate the highest revenue for them. In unconstrained conditions, various types of products can be offered, but in constrained conditions, there exists a maximum limit on the quantity of products. They addressed the fact that even unconstrained problems are hard. To develop a framework for solving the model, the problem was transformed into a corresponding problem of locating a function's fixed point, however, determining the function's value at each point necessitated addressing a nonlinear integer problem. Qiu et al. (2020) delved into the examination of how heterogeneous brands can be incorporated into assortment planning. They utilized a structure to model the customer choice process, where customers first select their desired brand and then choose products associated with that brand. They formulated this problem through a dynamic, time-constrained planning approach."

Bijmolt et al. (2021) investigated the factors influencing demand within the context of Omnichannel and identified product assortment, inventory levels, distribution and delivery, and returns as significant contributing factors. Wu and Pei (2023) conducted a comprehensive review of models employed for solving the product assortment selection problem and categorized the models used in the studies. This categorization is based on customer choice models, ranking-based models such as Multinomial Logit (MNL) models, and the models of external demand.

Hübner et al. (2022) conducted a literature review in the field of supply chain to identify influential tools for enhancing the performance and profitability of retailers. They ultimately highlighted assortment planning as one of the five key indicators for improving the sales level of retailers. Sajadi and Ahmadi (2022) concentrated on creating a unified mathematical model to optimize product assortment planning, shelf space allocation, and inventory control for perishable goods. This model aims to maximize sales and profitability while taking into account supplier costs, ordering processes, assortment strategies, holding costs, and procurement expenses.

Jasin et al. (2023) addressed a problem where customers might opt for purchasing a bundle of products rather than individual items. In their study, they considered the challenges of assortment planning, both with and without capacity, using the Multinomial Logit with Multiple Variables (MVMNL) model, recognized as one of the most widely used models for multi-variable choice Across marketing and empirical research literature. The analyses in this research demonstrate that overlooking multi-item purchasing behavior in assortment determination can significantly impact retailers' profitability, emphasizing its practical importance in retailing.

Kim et al. (2021) presented a model for maximizing the profits of retailers that involved decision-making regarding product selection and reshuffling with the allocation of shelf spaces. They incorporated factors influencing customer decisions in their model, accounting for space elasticity, cross-space effects, and the effects of position stability. This article's examination was based on a two-dimensional representation, where all shelves and products have width and height. The demand function in this model is non-convex, which is why they employed a mixed-integer non-linear model for solving it, using two heuristic algorithms - the Tabu Search and Genetic algorithms.

Table 1 offers an overview of recent research in the realm of promotion and assortment planning, emphasizing the interaction between these two areas.

Table 1
A Review of Recent Studies in the Field of Promotion and Assortment Planning

Article	Assortment	Promotion	Summary
Singh et al. (2013)	√		conducted an investigation into the relationship between product assortment and sales growth.
Greenstein-Messica et al. (2020)		√	Examined the relationship between Promotion and sales growth in the context of e-commerce.
Qiu et al. (2021)	√		Explored the impact of assortment planning on customer choices.
Bijmolt et al. (2021)	√		Studied the impact of factors such as product assortment, inventory levels, distribution, and returns on sales.
Ilyas et al. (2022)		√	Explored the relationship between Promotion and customer satisfaction
Hamdani (2022)		√	Examined the relationship between Promotion and customer retention.
Mohammadi-Pour et al. (2023)		√	Studied the relationship between Promotion and sales growth.
This study	√	√	Investigated the simultaneous impact of Promotion and assortment planning on final sales

2.3. Methods for solving problems

A variety of heuristic methods have been used in the literature. As an example, McElreath et al. (2010) compared three methods for solving the assortment problem: GA, TS, and simulated annealing (SA). Liao et al. propose a rough simulation-hybrid GA (2017) for the problem of substitution in multi-period and multi-product assortment planning.

We employ techniques from the nonlinear and integer optimization literature in this paper. This paper uses a mixed-integer nonlinear program (MINLP) to solve the assortment-promotion optimization problem. These MINLPs are computationally extremely challenging due to the high nonlinearity of the utility and demand functions under consideration. Hemmecke et al. (2010) have demonstrated that under specific structural conditions, there exist polynomial-time algorithms for solving Mixed-Integer Nonlinear Programming (MINLP) problems. However, as noted by Grossmann (2002), many MINLPs do not meet these conditions and, therefore, require alternative solution methods such as outer approximation, branch and bound, extended cutting planes and generalized Benders.

Here are some of the main contributions we made:

1. This is the first-time assortment and promotion have been considered together. There is a strong relationship between these two areas, so not paying attention to one of them will cause you to make incorrect decisions. As an example, when we look at the optimal combination of products, the final price of the product (after promotional discounts) affects the optimization of the product portfolio and its demand. Is it possible to determine the optimal product portfolio without considering it? The opposite is also true. Choosing the best promotion depends a lot on what products are placed together. Here, we have simultaneously addressed these two issues for the first time, and we have filled a gap in this regard.
2. The proposed nonlinear model was converted into a linear model using a special technique. This paper presents a new approach to a class of optimization problems in sales promotion and assortment that can be widely employed in these environments.

3. Model Description

Consider a market where stores compete for sales of products belonging to different categories. There is a chain with S stores in this competitive market. Additionally, these stores offer L different types of products and J items for each product type. For product j, there are I brands to choose from. We need to specify the products and brands that should be used for each store. A sample assortment scenario is shown in Fig. 1.

Fig 1. Variations in assortment scenarios

It can be seen that both scenarios have fruit juice on the second floor (from above), however, in scenario 1, there are two different SKUs (brands) and in scenario 2, there is one. In this scenario, one SKU has been removed and its vacancy is filled with SKU 1. The same thing occurs on other floors as well. A good assortment involves selecting the right combination of products and brands of each item. The disregard for the importance of this matter can lead to significant sales loss. To illustrate this point further, consider the previous example where a limited number of brands offered for a product can lead to a reduction in sales. This is because a greater variety of brands can attract a wider range of customers. Additionally, if a customer fails to find one of the items they need in a store, they may refrain from purchasing other items on their shopping list from that store, ultimately reducing the store's overall profit. For instance, a customer who visits a store to buy a specific type of Bologna, sauce, and a beverage, and if that specific Bologna is not available, they might abstain from buying the other items, resulting in not only a loss in the profit from selling the shirt but also the profits from the sale of the beverage and sauce. On the other hand, it's also possible that when a store only provides one brand of a particular item, given the substitutability of goods, a portion of the market share of two rival brands could be added to the sales of this item. If the profit margin of this item is higher in comparison to the two rival brands, the overall profit of the seller may increase.

Moreover, we should determine which promotion discount is optimal for each product in the portfolio over the planning period. The neglect of this matter entails various adverse effects. One of the primary effects of not providing optimal promotions for store products is a decrease in sales. Optimal promotions incentivize customers to purchase and use the desired products. Failing to offer suitable promotions can deter customers from making purchases, resulting in reduced sales. A decrease in sales, as a direct consequence of not offering optimal promotions, leads to a decrease in the store's profitability. This is naturally of critical importance in financial management and business sustainability. The presentation of optimal promotions for products is considered a fundamental element in enhancing the brand image. Successful promotions create a positive perception among customers regarding the brand and its products. Failing to provide appropriate promotions can weaken the brand image and the credibility of the store. Furthermore, improper promotion offerings, despite increasing sales, may lead to a reduction in the store's profitability, to the extent that in the absence of promotions, the store's profit may be higher compared to when promotions are implemented.

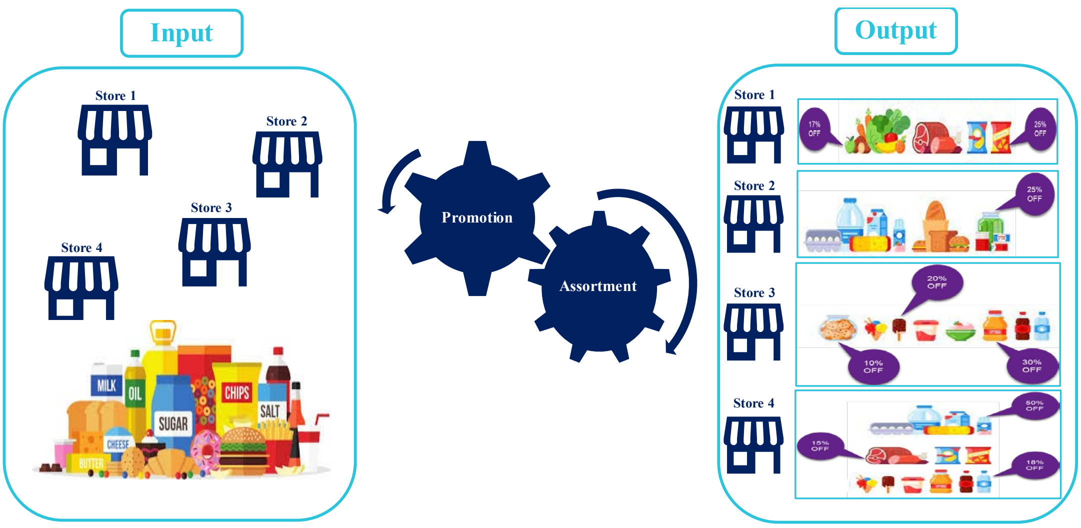

Fig 2. Product Assortment Presented in Each Store with Applied Promotions.

For a better understanding of the research problem, Fig. 2 is presented. As evident, inputs of the model consists of available products with different brands and suppliers, and the chain store manager utilizes the model to decide which products go into each store and the extent to which each product should be promoted during each period. Consequently, the optimal product assortment and the level of promotion for each product are determined in different time periods.

Through the entire paper, the following symbols will be employed:

Indices:

$l, \acute{l}$	Product type	$l, \acute{l}$=(1,2,…, L)
$j, \acute{j}$	Item	$j, \acute{j}$ =(1,2,…, J)
$i, \acute{i}$	Brand	$i, \acute{i}$ =(1,2,…, I)
$t, \acute{t}$	Period	$t, \acute{t}$ =(1,2,…, T)
s	Store number	s =(1,2,…, S)
k	Subsidiary for the previous Periods	k =(1,2,…, K)
v	Promotion interval	v =(1,2,…, V)

Parameters:

AM_{slj}	Maximum number of brands allowed for each product lj at Store s
Ao_{sl}	Maximum number of items for each product type l at Store s
M	Big number
IP_{lji}	Initial price of each SKU lji (toman)
V_v	Allowed numbers for promotion amount
$MaxPr_t$	Maximum number of promotions per each period t
$MaxP_{lji}$	Maximum value (%) of promotion for each SKU lji
D_{sljit}	Initial demand number of each SKU lji in each period t at Store s
α_{lji}	Effect of promotion (%) of each SKU lji on increasing sales of each SKU lji
β_{lji}	Effect of saving (%) SKU lji in the sale of the SKU lji in further periods
$\gamma_{lji\acute{l}\acute{j}\acute{i}}$	Effect of saving (%) SKU $\acute{l}\acute{j}\acute{i}$ in the sale of the SKU lji in further periods
$E_{lji\acute{l}\acute{j}\acute{i}}$	Effect of promotion (%) of each SKU $\acute{l}\acute{j}\acute{i}$ on the sale of other SKU lji
$\delta_{lji\acute{l}\acute{j}\acute{i}}$	Effect of the absence (%) of SKU $\acute{l}\acute{j}\acute{i}$ on the sale of SKU lji
C_{lji}	Purchase price of SKU lji (toman)

Decision Variable:

A_{slji}	1 If SKU lji was in the product portfolio of store s in period t; 0, otherwise
S_{sljit}	Number of SKU lji sales in period t at Store s
Pr_{sljit}	Value (%) of SKU lji promotion in period t at Store s
W_{vsljit}	1 if SKU lji is on promotion at amount of V_v in period t; 0, otherwise
R_{ljit}	1 If SKU lji is on promotion in period t; 0, otherwise
TPr_{ljit}	Final value (%) of SKU lji promotion in period t at Store s
B_{sljit}	1 If the sales of SKU lji in period t at Store s are more than the initial demand of SKU lji; 0, otherwise
P_{sljit}	Selling price of SKU lji in period t at Store s (toman)

Chain profits are calculated by multiplying total sales by profit per unit sold as follows (P1):

$$\text{Max } Z = \sum_{s=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{t=1}^{T}(S_{sljit} \times (P_{sljit} - C_{ljit})) \tag{1}$$

Eq. (1) calculates the profit from each SKU sold in each store during various periods. The constraints in the model and its explanations are as follows:

Constraints

$$\sum_{i=1}^{I} A_{slji} \leq AM_{slj} \qquad \forall\, s, l, j \tag{2}$$

$$\sum_{j=1}^{J}\sum_{i=1}^{I} A_{slji} \leq Ao_{sl} \qquad \forall\, s, l \tag{3}$$

$$S_{sljit} \leq A_{slji} \times M \qquad \forall\, s, l, j, i, t \tag{4}$$

$$Pr_{sljit} \leq A_{slji} \times M \qquad \forall\, s, l, j, i, t \tag{5}$$

$$Pr_{sljit} \leq TPr_{ljit} \qquad \forall\, s, l, j, i, t \tag{6}$$

$$Pr_{sljit} \geq TPr_{ljit} - M \times (1 - A_{slji}) \qquad \forall\, s, l, j, i, t \tag{7}$$

$$P_{sljit} = IP_{lji} \times (1 - Pr_{sljit}) \qquad \forall s, l, j, i, t \tag{8}$$

$$Pr_{sljit} = \sum_{v=1}^{V} (W_{vsljit} \times V_v) \qquad \forall s, l, j, i, t \tag{9}$$

$$\sum_{v=1}^{V} W_{vsljit} \leq 1 \qquad \forall s, l, j, i, t \tag{10}$$

$$TPr_{ljit} \leq R_{ljit} \times M \qquad \forall\, l, j, i, t \tag{11}$$

$$R_{ljit} \leq TPr_{ljit} \times M \qquad \forall\, l, j, i, t \tag{12}$$

$$\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{i=1}^{I} R_{ljit} \leq MaxPr_t \qquad \forall\, t \tag{13}$$

$$TPr_{ljit} \leq MaxP_{lji} \qquad \forall\, l, j, i, t \tag{14}$$

$$\begin{aligned} S_{sljit} \leq\; & D_{sljit} + \left(\alpha_{lji} \times TPr_{ljit} \times D_{sljit}\right) \\ & - \sum_{k=1}^{t-1} \left(\left(S_{slji(t-k)} - D_{slji(t-k)}\right) \times \beta_{lji} \times B_{slji(t-k)} \times \left(\frac{1}{2}\right)^{k-1} \right) \\ & - \sum_{k=1}^{t-1} \sum_{\acute{l} \neq l} \sum_{\acute{j} \neq j} \sum_{\acute{i} \neq i} \left(\left(S_{s\acute{l}\acute{j}\acute{i}(t-k)} - D_{s\acute{l}\acute{j}\acute{i}(t-k)}\right) \times \gamma_{lji\acute{l}\acute{j}\acute{i}} \times B_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \left(\frac{1}{2}\right)^{k-1} \right) \\ & + \left(\sum_{\acute{l} \neq l} \sum_{\acute{j} \neq j} \sum_{\acute{i} \neq i} E_{lji\acute{l}\acute{j}\acute{i}} \times Pr_{sljit} \times D_{s\acute{l}\acute{j}\acute{i}t} \right) \\ & + \sum_{\acute{l} \neq l} \sum_{\acute{j} \neq j} \sum_{\acute{i} \neq i} \left(\left(1 - A_{s\acute{l}\acute{j}\acute{i}}\right) \times D_{s\acute{l}\acute{j}\acute{i}t} \times \delta_{lji\acute{l}\acute{j}\acute{i}} \right) \end{aligned} \qquad \forall\, s, l, j, i, t \tag{15}$$

$$S_{sljit} - D_{sljit} \leq B_{sljit} \times M \qquad \forall s, l, j, i, t \tag{16}$$

$$(B_{sljit} - 1) \times M \leq S_{sljit} - D_{sljit} \qquad \forall\, s, l, j, i, t \tag{17}$$

For each SKU, constraint (2) limits the number of brands. In constraint (3), there is a limit on how many items are allowed per type of product. Constraint (4) prohibits the sale of products that are not selected in the product assortment. In accordance with constraint (5), products not included in the product assortment cannot be promoted. A unique promotion discount will be applied to all stores due to constraints (6) and (7). Constraints (8) depicts the price equation for each SKU. As shown in constraints (9) and (10), there are a set of limitations on how many products can be promoted at a time. In each period, the number of promotions is limited by constraints (11)-(13). Each SKU is limited to a certain number of promotions within a period by constraint (14). Sales are limited by constraints (15). The first term of the right-hand side of relation (15) displays the initial demand for each SKU. Each SKU's promotion has an effect on its sales in the second term. The third term shows how previous stock levels affect sales of each SKU. Term four shows how stocks of other SKUs have affected sales of each SKU in the past. The fifth term displays how other SKUs' promotions affect the sale of each SKU and finally the sixth term demonstrates how each SKU's sales are affected by the absence of other SKUs. It is stated in constraints (16) and (17) that the stocking of a product becomes meaningful only if its sales exceed its initial demand.

In the context of our problem, we are confronted with an integer nonlinear programming Issue. The following Part introduces two approaches for solving this problem. Firstly, we reframe it as a mixed integer linear programming problem, and subsequently, a heuristic approach is presented.

4. Solution Methods

4.1 Integer linear formulation

The nonlinear problem can be solved by substituting artificial variables for quadratic terms. Three steps can be taken to linearize the problem. Their explanations are listed below.

Step 1: Linearization of $\left(\left(\boldsymbol{S}_{\boldsymbol{slji(t-k)}} - \boldsymbol{D}_{\boldsymbol{slji(t-k)}}\right) \times \boldsymbol{\beta}_{\boldsymbol{lji}} \times \boldsymbol{B}_{\boldsymbol{slji(t-k)}} \times \left(\frac{1}{2}\right)^{\boldsymbol{k-1}}\right)$ in relation 15:

We have

$$\left(\left(S_{slji(t-k)} - D_{slji(t-k)}\right) \times \beta_{lji} \times B_{slji(t-k)} \times \left(\frac{1}{2}\right)^{k-1}\right) = (S_{slji(t-k)} \times \beta_{lji} \times B_{slji(t-k)} \times \left(\frac{1}{2}\right)^{k-1}) - (D_{slji(t-k)} \times \beta_{lji} \times B_{slji(t-k)} \times \left(\frac{1}{2}\right)^{k-1}) \quad (18)$$

Let a variable $S_{slji(t-k)} \times \beta_{lji} \times B_{slji(t-k)} \times \left(\frac{1}{2}\right)^{k-1} = H_{slji(t-k)}$.

There are the following inequalities:

$$H_{slji(t-k)} \leq S_{slji(t-k)} \times \beta_{lji} \times \left(\frac{1}{2}\right)^{k-1}$$

$$H_{slji(t-k)} \leq B_{slji(t-k)} \times M$$

$$H_{slji(t-k)} \geq S_{slji(t-k)} \times \beta_{lji} \times \left(\frac{1}{2}\right)^{k-1} - M \times (1 - B_{slji(t-k)})$$

Step 2: Linearization of $\left(\boldsymbol{S}_{\boldsymbol{s\acute{l}\acute{j}\acute{i}(t-k)}} - \boldsymbol{D}_{\boldsymbol{s\acute{l}\acute{j}\acute{i}(t-k)}}\right) \times \boldsymbol{\gamma}_{\boldsymbol{lji\acute{l}\acute{j}\acute{i}}} \times \boldsymbol{B}_{\boldsymbol{s\acute{l}\acute{j}\acute{i}(t-k)}} \times \left(\frac{1}{2}\right)^{\boldsymbol{k-1}}$ in relation 15:

We have

$$\left(S_{s\acute{l}\acute{j}\acute{i}(t-k)} - D_{s\acute{l}\acute{j}\acute{i}(t-k)}\right) \times \gamma_{lji\acute{l}\acute{j}\acute{i}} \times B_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \left(\frac{1}{2}\right)^{k-1} = \left(S_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \gamma_{lji\acute{l}\acute{j}\acute{i}} \times B_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \left(\frac{1}{2}\right)^{k-1}\right) - (D_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \gamma_{lji\acute{l}\acute{j}\acute{i}} \times B_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \left(\frac{1}{2}\right)^{k-1}) \quad (19)$$

Assume a variable $S_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \gamma_{lji\acute{l}\acute{j}\acute{i}} \times B_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \left(\frac{1}{2}\right)^{k-1} = \acute{H}_{s\acute{l}\acute{j}\acute{i}(t-k)}$.

Therefore, we have the following inequalities:

$$\acute{H}_{s\acute{l}\acute{j}\acute{i}(t-k)} \leq S_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \gamma_{lji\acute{l}\acute{j}\acute{i}} \times \left(\frac{1}{2}\right)^{k-1}$$

$$\acute{H}_{s\acute{l}\acute{j}\acute{i}(t-k)} \leq B_{s\acute{l}\acute{j}\acute{i}(t-k)} \times M$$

$$\acute{H}_{s\acute{l}\acute{j}\acute{i}(t-k)} \geq \left(S_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \gamma_{lji\acute{l}\acute{j}\acute{i}} \times \left(\frac{1}{2}\right)^{k-1}\right) - M \times \left(1 - B_{s\acute{l}\acute{j}\acute{i}(t-k)}\right)$$

Step 3: Linearization of $\boldsymbol{S}_{\boldsymbol{sljit}} \times \left(\boldsymbol{P}_{\boldsymbol{sljit}} - \boldsymbol{C}_{\boldsymbol{ljit}}\right)$ in relation 1:

We have

$$S_{sljit} \times \left(P_{sljit} - C_{ljit}\right) = S_{sljit} \times \left(\left(IP_{lji} \times \left(1 - Pr_{sljit}\right)\right) - C_{ljit}\right) = \left(S_{sljit} \times IP_{lji}\right) - \left(S_{sljit} \times IP_{lji} \times Pr_{sljit}\right) - \left(S_{sljit} \times C_{ljit}\right) \quad (20)$$

For expression $\left(S_{sljit} \times IP_{lji} \times Pr_{sljit}\right)$ we have $\left(S_{sljit} \times IP_{lji} \times Pr_{sljit}\right) = S_{sljit} \times IP_{lji} \times \sum_{v=1}^{V}\left(W_{vsljit} \times V_v\right) = \sum_{v=1}^{V}\left(W_{vsljit} \times V_v \times S_{sljit} \times IP_{lji}\right)$.

Assume a variable $W_{vsljit} \times V_v \times S_{sljit} \times IP_{lji} = H''_{vsljit}$.

As a result, there are the following inequalities:

$$H''_{vsljit} \leq V_v \times S_{sljit} \times IP_{lji}$$

$$H''_{vsljit} \leq W_{vsljit} \times M$$

$$H''_{vsljit} \geq (V_v \times S_{sljit} \times IP_{lji}) - M \times (1 - W_{vsljit})$$

Ultimately, the problem can be reformulated as follows (P2):

$$\text{Max } Z = \sum_{s=1}^{S}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{t=1}^{T}\left(S_{sljit} \times IP_{lji}\right) - \left(\sum_{v=1}^{V} H''_{vsljit}\right) - \left(S_{sljit} \times C_{ljit}\right) \tag{21}$$

$$\sum_{i=1}^{I} A_{slji} \leq AM_{slj} \qquad \forall\, s, l, j \tag{22}$$

$$\sum_{j=1}^{J}\sum_{i=1}^{I} A_{slji} \leq Ao_{sl} \qquad \forall\, s, l \tag{23}$$

$$S_{sljit} \leq A_{slji} \times M \qquad \forall\, s, l, j, i, t \tag{24}$$

$$Pr_{sljit} \leq A_{slji} \times M \qquad \forall\, s, l, j, i, t \tag{25}$$

$$Pr_{sljit} \leq TPr_{ljit} \qquad \forall\, s, l, j, i, t \tag{26}$$

$$Pr_{sljit} \geq TPr_{ljit} - M \times (1 - A_{slji}) \qquad \forall\, s, l, j, i, t \tag{27}$$

$$P_{sljit} = IP_{lji} \times (1 - Pr_{sljit}) \qquad \forall s, l, j, i, t \tag{28}$$

$$Pr_{sljit} = \sum_{v=1}^{V} (W_{vsljit} \times V_v) \qquad \forall s, l, j, i, t \tag{29}$$

$$\sum_{v=1}^{V} W_{vsljit} \leq 1 \qquad \forall s, l, j, i, t \tag{30}$$

$$TPr_{ljit} \leq R_{ljit} \times M \qquad \forall\, l, j, i, t \tag{31}$$

$$R_{ljit} \leq TPr_{ljit} \times M \qquad \forall\, l, j, i, t \tag{32}$$

$$\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{i=1}^{I} R_{ljit} \leq MaxPr_t \qquad \forall\, t \tag{33}$$

$$TPr_{ljit} \leq MaxP_{lji} \qquad \forall\, l, j, i, t \tag{34}$$

$$\begin{aligned} S_{sljit} \leq\ & D_{sljit} + \left(\alpha_{lji} \times TPr_{ljit} \times D_{sljit}\right) \\ & - \sum_{k=1}^{t-1}\left(H_{slji(t-k)} - (D_{slji(t-k)} \times \beta_{lji} \times B_{slji(t-k)} \times \left(\frac{1}{2}\right)^{k-1})\right) \\ & - \sum_{k=1}^{t-1}\sum_{\acute{l}\neq l}\sum_{\acute{j}\neq j}\sum_{\acute{i}\neq i}\left(\acute{H}_{s\acute{l}\acute{j}\acute{i}(t-k)} - (D_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \gamma_{lji\acute{l}\acute{j}\acute{i}} \times B_{s\acute{l}\acute{j}\acute{i}(t-k)} \times \left(\frac{1}{2}\right)^{k-1})\right) \\ & + \left(\sum_{\acute{l}\neq l}\sum_{\acute{j}\neq j}\sum_{\acute{i}\neq i} E_{lji\acute{l}\acute{j}\acute{i}} \times Pr_{sljit} \times D_{s\acute{l}\acute{j}\acute{i}t}\right) \\ & + \sum_{\acute{l}\neq l}\sum_{\acute{j}\neq j}\sum_{\acute{i}\neq i}\left(\left(1 - A_{s\acute{l}\acute{j}\acute{i}}\right) \times D_{s\acute{l}\acute{j}\acute{i}t} \times \delta_{lji\acute{l}\acute{j}\acute{i}}\right) \end{aligned} \qquad \forall\, s, l, j, i, t \tag{35}$$

$$S_{sljit} - D_{sljit} \leq B_{sljit} \times M \qquad \forall s, l, j, i, t \tag{36}$$

$$(B_{sljit} - 1) \times M \leq S_{sljit} - D_{sljit} \qquad \forall\, s, l, j, i, t \tag{37}$$

$$H_{slji(t-k)} \leq S_{slji(t-k)} \times \beta_{lji} \times \left(\frac{1}{2}\right)^{k-1} \qquad \forall\, s, l, j, i, t \tag{38}$$

$$H_{slji(t-k)} \leq B_{slji(t-k)} \times M \qquad \forall\, s,l,j,i,t \quad (39)$$

$$H_{slji(t-k)} \geq S_{slji(t-k)} \times \beta_{lji} \times \left(\frac{1}{2}\right)^{k-1} - M \times (1 - B_{slji(t-k)}) \qquad \forall\, s,l,j,i,t \quad (40)$$

$$\acute{H}_{s\acute{l}j\acute{i}(t-k)} \leq S_{s\acute{l}j\acute{i}(t-k)} \times \gamma_{lji\acute{l}j\acute{i}} \times \left(\frac{1}{2}\right)^{k-1} \qquad \forall\, s,l,j,i,t \quad (41)$$

$$\acute{H}_{s\acute{l}j\acute{i}(t-k)} \leq B_{s\acute{l}j\acute{i}(t-k)} \times M \qquad \forall\, s,l,j,i,t \quad (42)$$

$$\acute{H}_{s\acute{l}j\acute{i}(t-k)} \geq \left(S_{s\acute{l}j\acute{i}(t-k)} \times \gamma_{lji\acute{l}j\acute{i}} \times \left(\frac{1}{2}\right)^{k-1}\right) - M \times \left(1 - B_{s\acute{l}j\acute{i}(t-k)}\right) \qquad \forall\, s,l,j,i,t \quad (43)$$

$$H''_{vsljit} \leq V_v \times S_{sljit} \times IP_{lji} \qquad \forall\, s,l,j,i,t \quad (44)$$

$$H''_{vsljit} \leq W_{vsljit} \times M \qquad \forall\, s,l,j,i,t \quad (45)$$

$$H''_{vsljit} \geq (V_v \times S_{sljit} \times IP_{lji}) - M \times (1 - W_{vsljit}) \qquad \forall\, s,l,j,i,t \quad (46)$$

which the resulting mixed integer linear programming (MIP) formulation can be directly tackled by MIP solvers. However, for large-scale issues, achieving global optimality within a reasonable computational time can be challenging.

4.2 Discrete Firefly Algorithm (DFA)

When exact methods are too time consuming due to the size of the problem, heuristic methods are devised to implement procedures and to obtain approximate solutions. In this study, a comparison of results from various metaheuristic methods, such as Grey Wolf and Firefly algorithms, demonstrated that the Firefly algorithm provides superior results, closer to the optimal solution. Firefly meta-heuristics are inspired by the social behavior and bioluminescent communication of fireflies (Yang, 2009) and are used for solving optimization problems (Yang, 2009). In Firefly Algorithm (FA), the modification of light intensity and the formulation of attractiveness are considered two crucial aspects.

The use of FA in engineering spans a wide range of areas including tower structures design, system identification design, competitive location design, power system design, antenna design, and reliability analysis for construction systems. Further details can be found in Tilahun and Ngnotchouye (2017).

The three rules for designing an algorithm inspired by fireflies are:

1. Due to the homogeneity of fireflies, they are all attracted to other fireflies regardless of their gender.
2. As distance between fireflies increases, the attractiveness decreases because of the decrease in brightness. There is a difference in the amount of light traveling to the brighter firefly for both flashing fireflies. In the absence of a brighter firefly, this individual wanders randomly across the search space.
3. A firefly's brightness is determined or influenced by its objective function.

For optimization problems aimed at maximizing, it's typical to relate a firefly's brightness to the value of the objective function. The update equation for a pair of fireflies, denoted as z and y, is given by:

$$X_z^{t+1} = X_z^t + \beta_0 e^{-\gamma r_{zy}^2}\left(X_z^t - X_y^t\right) + \alpha \varepsilon_z^t \quad (47)$$

The first term represents the step size, The second term accounts for attraction, while the third term represents randomization. Cartesian distance or gaussian distance can be used to measure distance between fireflies.

We use discrete firefly to obtain promotion and assortment variables in this paper.

4.2.1 Pseudo code

The stages of the algorithm DFA can be succinctly condensed using the pseudo code provided in the Fig. 3.

Run DFA
Initialize the population of fireflies with their initial positions X_z $(z = 1,2, \dots, n)$.
Define the objective function, which calculates the light intensity I_z at position X_z.
Assign the values for the light absorption coefficient γ, the randomization parameter α, and the maximum number of iterations (MItr).
when (t <MItr)

for z = 1: n all fireflies
for y = 1: z
if $(I_y > I_z)$, Shift firefly z towards firefly y by adjusting its position in each dimension.
Attractiveness varies with distance r via exp $[-\gamma r^2]$ for promotion and assortment of new store.

$$X_z = X_z + \beta_0 e^{-\gamma r^2}(X_y - X_z) + \alpha\varepsilon_z$$

Discrete the promotion variable of z-th firefly.

$$S(X_{zys}^t) = \frac{1}{1 + \exp\,(-X_{zys}^t)}$$

Each firefly selects a discount promotion and Choose new assortment according on its variations of probabilities.
Evaluate new solution (position of z-th firefly) and update light intensity I_z.
end if
end for y
end for z
Determine the best firefly by comparing their values.
End while
Display the best-known solution found so far, along with its corresponding objective value.

Fig. 3. Steps of the DFA algorithm.

5 Computational examples

This part outlines the computational experiments conducted that were performed to assess the effectiveness of the model. Initially, a small-scale problem was solved using DICOPT, a MINLP solver available in GAMS, and the results were carefully examined. To showcase the efficiency and effectiveness of the proposed method, various examples of differing sizes were also solved. The computational experiments are executed on a system equipped with an Intel Core i7 processor running at 3.9 GHz and 16 GB of memory. The heuristic implementation is carried out using MATLAB R2022b.

5.1 An illustrative example

In a market, there are 4 stores. Suppose that there are 2 product types available in these stores, each with 3 products and 4 brands. Customer initial demand for various SKUs in each store on 3 periods is shown in Table 2.

Table 2
Demand, Initial Price and Purchase cost of different SKUs

			Period 1				Period 2				Period 3				IP_{lji}	C_{lji}
			Str 1	Str 2	Str 3	Str 4	Str 1	Str 2	Str 3	Str 4	Str 1	Str 2	Str 3	Str 4		
Product Type 1	Pr 1	B 1	65	47	108	10	59	35	103	7	51	13	60	7	23	15
		B 2	95	22	130	38	56	7	72	10	87	13	68	12	23	12
		B 3	2	1	4	76	1	0	1	67	1	1	1	29	26	14
		B 4	84	78	108	68	52	46	51	61	38	69	2	53	26	17
	Pr 2	B 1	30	15	53	52	13	0	24	10	3	11	49	28	21	12
		B 2	75	70	144	30	12	37	92	11	66	7	26	23	22	16
		B 3	77	22	86	66	63	2	39	23	31	16	45	15	25	12
		B 4	35	18	35	7	23	3	35	4	7	9	0	5	24	13
	Pr 3	B 1	90	86	156	77	88	22	110	70	68	69	145	54	21	13
		B 2	34	22	34	11	21	16	0	9	27	17	26	8	24	20
		B 3	8	6	14	52	5	2	5	1	8	1	8	34	20	13
		B 4	58	33	86	16	56	20	2	3	49	15	16	5	25	13
Product Type 2	Pr 1	B 1	96	54	133	42	64	37	91	14	17	15	116	16	30	18
		B 2	68	39	85	8	2	16	25	3	48	22	47	4	21	15
		B 3	41	24	55	52	30	22	5	46	14	6	17	2	29	18
		B 4	11	2	14	32	10	2	5	13	8	2	9	29	26	15
	Pr 2	B 1	93	55	155	4	21	5	139	1	28	16	53	0	20	20
		B 2	38	22	40	48	6	7	11	15	3	8	22	10	27	13
		B 3	63	23	82	56	18	7	29	20	8	19	8	19	27	12
		B 4	67	43	108	89	33	4	70	1	50	35	53	71	22	15
	Pr 3	B 1	16	9	22	51	2	3	18	10	0	2	19	4	21	16
		B 2	85	28	129	20	55	13	57	2	51	5	43	19	27	18
		B 3	3	2	3	26	0	1	1	0	1	0	2	22	28	19
		B 4	15	5	15	92	13	3	12	82	2	5	8	25	30	16

Table 3 provides other information about products.

Table 3
Other information about products

			AM_{sli}				Ao_{sl}				α_{lji}	β_{lji}	$MaxP_{lji}$
			Str 1	**Str 2**	**Str 3**	**Str 4**	**Str 1**	**Str 2**	**Str 3**	**Str 4**			
Product Type 1	**Pr 1**	**B 1**	4	3	3	2	9	5	5	8	1.03	1.12	31%
		B 2									1.05	1.18	36%
		B 3									1.05	1.06	29%
		B 4									1.02	1.15	35%
	Pr 2	**B 1**	3	2	1	3					1.05	1.02	10%
		B 2									1.03	1.03	20%
		B 3									1.02	1.15	23%
		B 4									1.10	1.07	15%
	Pr 3	**B 1**	2	4	2	3					1.09	1.14	39%
		B 2									1.07	1.04	28%
		B 3									1.07	1.19	26%
		B 4									1.04	1.14	14%
Product Type 2	**Pr 1**	**B 1**	2	4	4	2	5	8	9	5	1.07	1.01	1%
		B 2									1.09	1.08	36%
		B 3									1.03	1.15	21%
		B 4									1.03	1.04	8%
	Pr 2	**B 1**	1	4	3	2					1.06	1.05	9%
		B 2									1.08	1.16	34%
		B 3									1.07	1.14	31%
		B 4									1.00	1.11	9%
	Pr 3	**B 1**	4	4	2	3					1.07	1.09	5%
		B 2									1.06	1.05	10%
		B 3									1.06	1.05	7%
		B 4									1.06	1.05	29%

In accordance with the prevailing scenario in other articles, the determination of the optimal product assortment and the optimal discount amount is considered in two separate consecutive stages. Based on the process outlined in Fig. 4, the optimal product assortment is first determined in the first step, and then, based on the specified product assortment, the proposed discount amount for each product is determined. Accordingly, in the first stage, the discount amount is introduced as a parameter with zero value, and the optimal product assortment is determined. In the next stage, the optimal product assortment determined in the first stage is introduced as a parameter in the second-stage model, and the Promotion amount in this stage is determined. Therefore, Table 5 indicates the optimal Promotion values for each of the products available in each store, as shown in Table 4.

Table 4
Optimal Assortment of the example

			A_{slji}			
			Str 1	**Str 2**	**Str 3**	**Str 4**
Product Type 1	**Pr 1**	**B 1**	1	1	1	1
		B 2	1	1	1	1
		B 3	1	0	0	0
		B 4	1	1	1	0
	Pr 2	**B 1**	1	1	0	0
		B 2	1	0	0	1
		B 3	1	1	1	1
		B 4	0	0	0	1
	Pr 3	**B 1**	0	0	0	1
		B 2	1	0	1	1
		B 3	1	0	0	0
		B 4	0	0	0	1
Product Type 2	**Pr 1**	**B 1**	0	1	1	0
		B 2	0	1	1	0
		B 3	0	1	0	1
		B 4	0	1	1	1
	Pr 2	**B 1**	0	1	1	0
		B 2	1	1	1	0
		B 3	0	0	0	0
		B 4	0	1	1	0
	Pr 3	**B 1**	1	0	0	1
		B 2	1	0	1	0
		B 3	1	1	1	1
		B 4	1	0	1	1

In Table 4, the optimal product assortment determined in each store is specified. Based on this, the presence or absence of each of the brands in each subgroup in each of the stores is respectively indicated by the values one and zero.

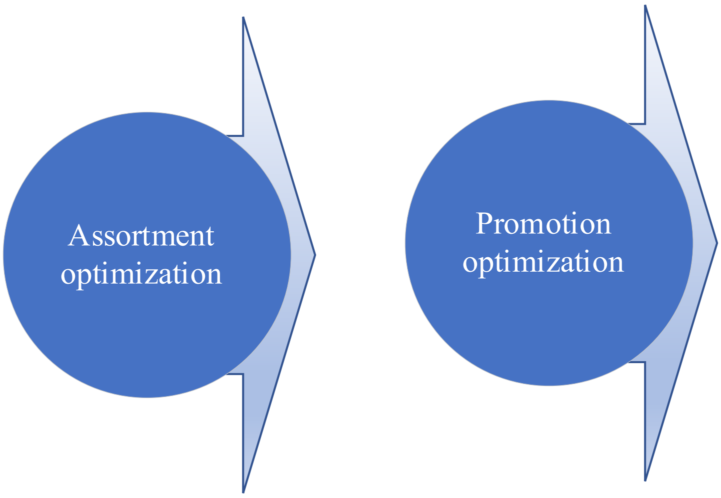

Fig 4. Prevailing scenario in other articles.

Table 5
Optimal Assortment of the example, scenario 1

			Period 1				Period 2				Period 3			
			Str 1	**Str 2**	**Str 3**	**Str 4**	**Str 1**	**Str 2**	**Str 3**	**Str 4**	**Str 1**	**Str 2**	**Str 3**	**Str 4**
Product Type 1	**Pr 1**	**B 1**	15%	15%	15%	15%	8%	8%	8%	8%	23%	23%	23%	23%
		B 2	24%	24%	24%	24%	34%	34%	34%	34%	23%	23%	23%	23%
		B 3	23%	-	-	-	1%	-	-	-	19%	-	-	-
		B 4	21%	21%	21%	-	18%	18%	18%	-	18%	18%	18%	-
	Pr 2	**B 1**	4%	4%	-	-	6%	6%	-	-	1%	1%	-	-
		B 2	5%	-	-	5%	7%	-	-	7%	4%	-	-	4%
		B 3	0%	0%	0%	0%	17%	17%	17%	17%	16%	16%	16%	16%
		B 4	-	-	-	14%	-	-	-	5%	-	-	-	2%
	Pr 3	**B 1**	-	-	-	28%	-	-	-	27%	-	-	-	25%
		B 2	7%	-	7%	7%	8%	-	8%	8%	10%	-	10%	10%
		B 3	0%	-	-	-	16%	-	-	-	13%	-	-	-
		B 4	-	-	-	3%	-	-	-	7%	-	-	-	4%
Product Type 2	**Pr 1**	**B 1**	-	1%	1%	-	-	1%	1%	-	-	1%	1%	-
		B 2	-	2%	2%	-	-	25%	25%	-	-	0%	0%	-
		B 3	-	12%	-	12%	-	9%	-	9%	-	8%	-	8%
		B 4	-	2%	2%	2%	-	3%	3%	3%	-	3%	3%	3%
	Pr 2	**B 1**	-	4%	4%	-	-	1%	1%	-	-	4%	4%	-
		B 2	15%	15%	15%	-	30%	30%	30%	-	15%	15%	15%	-
		B 3	-	-	-	-	-	-	-	-	-	-	-	-
		B 4	-	1%	1%	-	-	6%	6%	-	-	5%	5%	-
	Pr 3	**B 1**	4%	-	-	4%	2%	-	-	2%	5%	-	-	5%
		B 2	1%	-	1%	-	7%	-	7%	-	5%	-	5%	-
		B 3	1%	1%	1%	1%	0%	0%	0%	0%	2%	2%	2%	2%
		B 4	17%	-	17%	17%	13%	-	13%	20%	21%	-	20%	20%

The key differentiation in this study from previous research lies in the simultaneous determination of the two variables, the optimal product assortment and the discount level, whereas in earlier studies, these variables were determined independently and sequentially (Figure 4). This study emphasizes the necessity of simultaneously solving both variables, and the problem has been modeled and solved accordingly (Figure 5). As demonstrated by the provided example, the optimal discount level for each product is accessible in Table 5. dashes indicates that the product does not exist in the product assortment offered in that store. In this example, the profit of the retail chain store is equal to $24,114 for executing this scenario.

Fig 5. Dominant Scenario in this Study

Table 6
Optimal solution of the example scenario 2

			Period 1				Period 2				Period 3			
			Str 1	**Str 2**	**Str 3**	**Str 4**	**Str 1**	**Str 2**	**Str 3**	**Str 4**	**Str 1**	**Str 2**	**Str 3**	**Str 4**
Product Type 1	**Pr 1**	**B 1**	23%	23%	-	-	14%	14%	-	-	3%	3%	-	-
		B 2	35%	35%	35%	35%	17%	17%	17%	17%	20%	20%	20%	20%
		B 3	20%	-	20%	-	12%	-	12%	-	27%	-	27%	-
		B 4	28%	28%	28%	28%	4%	4%	4%	4%	10%	10%	10%	10%
	Pr 2	**B 1**	8%	8%	-	-	7%	7%	-	-	5%	5%	-	-
		B 2	18%	18%	-	18%	3%	3%	-	3%	19%	19%	-	19%
		B 3	16%	-	-	16%	21%	-	-	21%	17%	-	-	17%
		B 4	-	-	-	14%	-	-	-	5%	-	-	-	3%
	Pr 3	**B 1**	-	-	25%	25%	-	-	20%	20%	-	-	16%	16%
		B 2	28%	-	28%	28%	7%	-	7%	7%	17%	-	17%	17%
		B 3	20%	-	-	-	6%	-	-	-	4%	-	-	-
		B 4	-	-	-	3%	-	-	-	11%	-	-	-	11%
Product Type 2	**Pr 1**	**B 1**	-	1%	1%	-	-	1%	1%	-	-	1%	1%	-
		B 2	-	4%	4%	-	-	11%	11%	-	-	35%	35%	-
		B 3	-	21%	21%	21%	-	14%	14%	14%	-	2%	2%	2%
		B 4	-	5%	5%	5%	-	8%	8%	8%	-	8%	8%	8%
	Pr 2	**B 1**	-	8%	8%	-	-	6%	6%	-	-	2%	2%	-
		B 2	-	4%	4%	-	-	32%	32%	-	-	9%	9%	-
		B 3	-	12%	-	-	-	19%	-	-	-	9%	-	-
		B 4	7%	7%	7%	-	5%	5%	5%	-	7%	7%	7%	-
	Pr 3	**B 1**	2%	-	-	2%	2%	-	-	2%	5%	-	-	5%
		B 2	7%	-	7%	-	2%	-	2%	-	5%	-	5%	-
		B 3	4%	-	4%	4%	5%	-	5%	5%	5%	-	5%	5%
		B 4	13%	-	-	13%	26%	-	-	26%	8%	-	-	8%

The optimal solution for scenario 2 is presented in Table 6. The dashes indicate the absence of a product in the optimal assortment, and the values inside the table represent the optimal promotion levels. The profit of the chain store for simultaneously determining the optimal assortment and discount level for each product is equivalent to $31,411, resulting in a 30% increase in profit compared to the previous scenario. By comparing the profit obtained from implementing the dominant scenario with other articles and the current study, we conclude that the assumption of independence between these two key variables was a mistaken assumption. In other words, implementing the mentioned scenario leads to the allocation of unreasonable discounts for products without considering the effect of the product assortment and the impact of other products in the assortment, resulting in a significant loss in sales and, consequently, a reduction in the profitability of the chain store.

5.2 Sensitivity Analysis

In this part, modifications are applied to the parameters of the mentioned example, and then assess and analyze the percentage of variations in the optimal solutions.

5.2.1 Changes in substitution probability:

Table 7 presents the percentage of changes in the optimal solution, corresponding to variations in the substitution probability.

Table 7
The effect of changes in substitution probability

#	Effect of Absence	Number of Assortment Change						Total Promotion Change						Objective Function Change
		Product Type 1			Product Type 2			Product Type 1			Product Type 2			
		Pr1	Pr2	Pr3	Pr1	Pr2	Pr3	Pr1	Pr2	Pr3	Pr1	Pr2	Pr3	
1	0	-	-	-	-	-	-	-	-	-	-	-	-	-
2	0.2	1	1	-	2	3	-	5%	11%	10%	2%	3%	5%	4%
3	0.5	1	1	1	2	3	1	15%	21%	20%	14%	18%	25%	9%
4	0.8	2	2	2	3	3	2	27%	31%	24%	16%	22%	28%	15%
5	1	3	3	3	4	4	3	38%	42%	32%	19%	25%	35%	25%

Table 7 highlights the significant impact of the substitution probability on both the objective function and the optimal solution. Accurate estimation of this parameter is crucial to prevent making erroneous intentions. When the substitution probability is either zero or extremely low, the model behaves accordingly the complete product set state as there are no substitutable products. However, as the substitution probability increases, the probability of the complete product set state decreases, and the profit of the chain increases. Changes in the substitution probability not only affect the product assortment but also affect the optimal discounts of the stores.

5.2.2 Changes Demand:

In Table 8, we demonstrate changes in the optimal solution for some changes in the base customer demands.

Table 8
The effect of changes in demand

#	Change	Number of Assortment Change						Total Promotion Change						Objective Function Change
		Product Type 1			Product Type 2			Product Type 1			Product Type 2			
		Pr1	Pr2	Pr3	Pr1	Pr2	Pr3	Pr1	Pr2	Pr3	Pr1	Pr2	Pr3	
1	The main Problem	-	-	-	-	-	-	-	-	-	-	-	-	-
2	Increase 100% in one SKU	1	2	1	2	2	1	15%	21%	2%	14%	11%	22%	10%
3	Decrease 100% in one SKU	2	1	-	2	1	2	12%	15%	3%	10%	12%	4%	8%
4	Increase 20% for all	1	1	-	2	1	-	24%	12%	5%	10%	11%	5%	9%
5	Decrease 20% for all	1	-	1	1	-	1	25%	12%	8%	11%	18%	7%	11%

Table 8 illustrates the changes in the decision-making resulting from errors in estimating the base demand.

5.3 The test problems

This section includes multiple examples to showcase the performance of the proposed methods. The examples cover the following scenarios:

1. Nonlinear Problem: The MINLP solver (DICOPT) is utilized to solve a nonlinear problem.
2. Linearized Problem: The MIP solver (CPLEX-12.6.1.0) is employed to solve a linearized problem.
3. DFA Method: The DFA method is applied to address the given problem.

In each type of setting, ten problems have been generated, with parameters arbitrarily select from the following ranges: D_{sljit} ~ U(1, 100), IP_{lji} ~ U(11, 20), α_{lji} ~ U(1.00, 1.2), β_{lji} ~ U(1.00, 1.2), $E_{lji\acute{l}\acute{j}\acute{i}} \sim \mathrm{U}(0,1)$, $\gamma_{lji\acute{l}\acute{j}\acute{i}} \sim \mathrm{U}(0,1)$, $\delta_{lji\acute{l}\acute{j}\acute{i}} \sim \mathrm{U}(0,1)$, $C_{lji} \sim \mathrm{U}(1, 10)$.

5.3.1 The value of the DFA parameters

The γ should correspond to the scales of the design variables. One option is to use $\gamma = {}^{1}\!/_{\sqrt{L}}$ where L represents the average scale of the problem. The parameter is set to 0.5 after comparing the various values for it. For most cases, β0= 1 and α ∈ [0, 1] are considered. By comparing different values for α, the value 0.25 is suitable for this parameter. Based on the observations, it was found that in most cases, the optimal solution could be obtained after approximately 500 assessments. As a result, for the computational experiment, the decision was made to use 25 fireflies and conduct 20 generations.

5.3.2 The results

Our first step is to present detailed results for a chosen setting so we can explain how to generate the subsequent summary tables. Table 9 presents the results of 10 generated problems for s=10, l=15, j=5, i=2 and 100 DFA runs corresponding to 10 generated problems. There are two last lines showing the average and standard deviation for the whole group. The percentage difference between the best solutions obtained by DFA method and the optimal values obtained by optimization solvers, is shown in the Table 9. The column "Times found" shows how many times DFA found the best solution. MINLP, MIP, and DFA methods were also evaluated based on the CPU time spent in solving 10 generated problems.

Table 9
The variation in objectives and CPU time across ten examples, each with 10 stores, 15 product types, 5 products, and 2 brands per product

Case	Discrepancy in obj (%)	Number of similar answers	CPU seconds		
			DICOPT	CPLEX	DFA
1	0.532	8	1104.34	53.58	2.79
2	0.133	10	1193.63	53.53	3.83
3	0.533	7	1142.64	65.58	1.98
4	0.911	10	1147.29	56.18	3.80
5	0.471	10	1113.01	60.28	2.54
6	0.882	7	1131.44	69.60	3.57
7	0.794	11	1161.24	57.34	1.63
8	0.251	4	1119.15	65.78	1.03
9	0.964	15	1241.74	60.86	1.80
10	0.085	0	1248.86	58.35	1.68
Average	**0.555**	**8.3**	**1160.33**	**60.11**	**2.46**
Standard Deviation	**0.326**	**4.1**	**51.60**	**5.43**	**1.00**

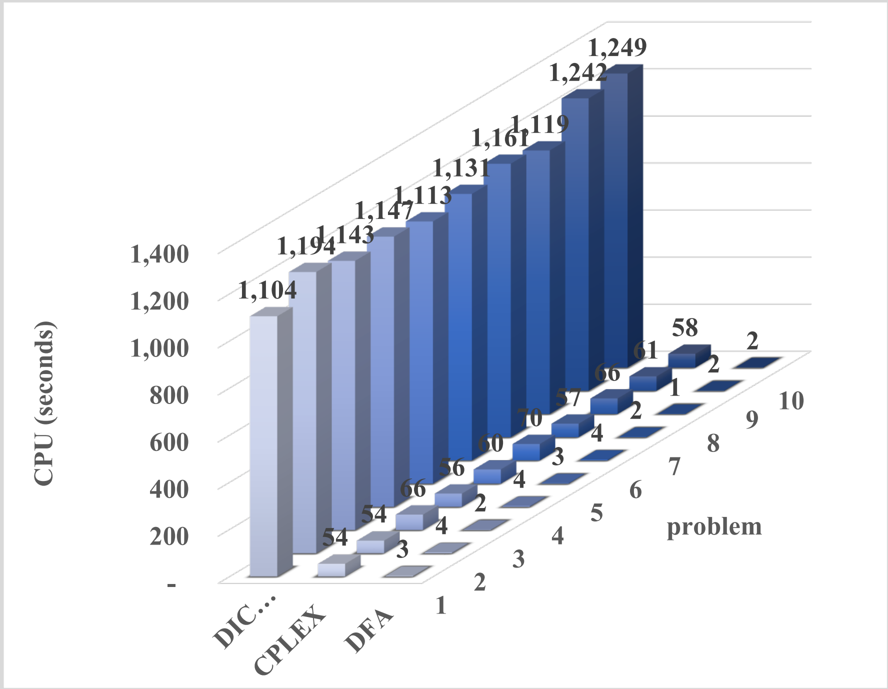

Fig. 6. Performance of three solution approaches from a time perspective

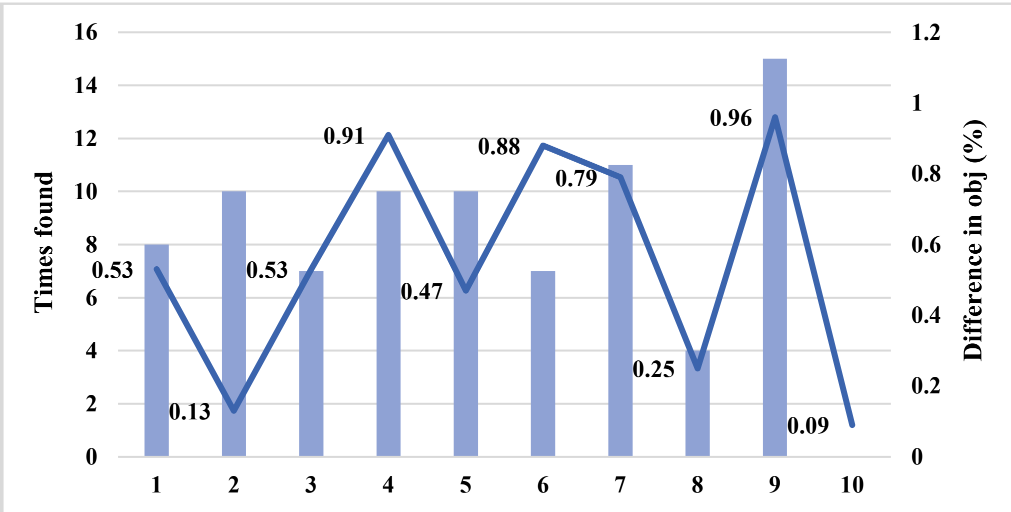

Fig. 7. changes in the objective function and the number of times the optimal solution was found in 10 Problems

Regarding the objective function, as illustrated in Table 9, DFA's solution is not much different from optimization solvers, although it can differ significantly from them in some cases. Additionally, it is worth noting that the DFA method exhibits significantly faster performance compared to the other two methods.

To check the results, only average values will be displayed from now on. A summary table will be generated based on the number of products. In it, each line corresponds to a table, such as Table 10. Those values are the averages of the solved instances, with standard deviations in brackets. In order to account for significant variations in the objective function differences within the set of 10 problems, we will also present the maximum difference observed. Additionally, the average difference across all settings will be provided in the last line, irrespective of the number of products present in each setting.

Table 10
The average and standard deviation values for the Discrepancies in objectives and CPU time for scenarios involving 10 stores, 15 product types, 2 brands, and 5 or 10 products

Products	Discrepancy in		CPU seconds		
	Obj (%)	Max (%)	DICOPT	CPLEX	DFA
5	0.56 (0.33)	0.97	1160.33 (51.60)	60.11 (5.43)	2.46 (1.00)
10	0.97 (0.79)	2.1	5251.81 (516.02)	92.41 (29.12)	3.25 (1.29)
All	**0.77 (0.56)**	**1.53**	**3206.07 (283.81)**	**76.26 (17.28)**	**2.86 (1.15)**

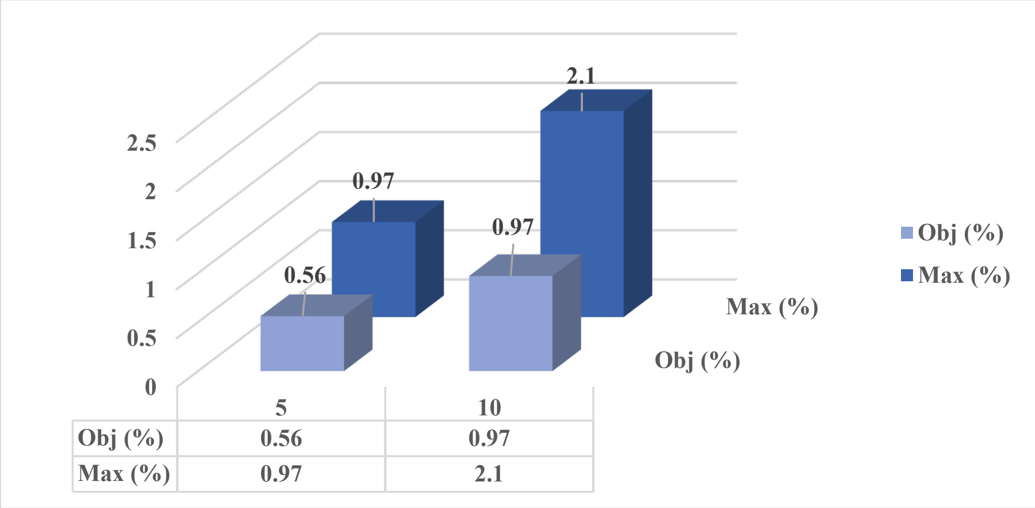

Fig 8. Percentage changes in Obj and Max resulting from variations in product groups

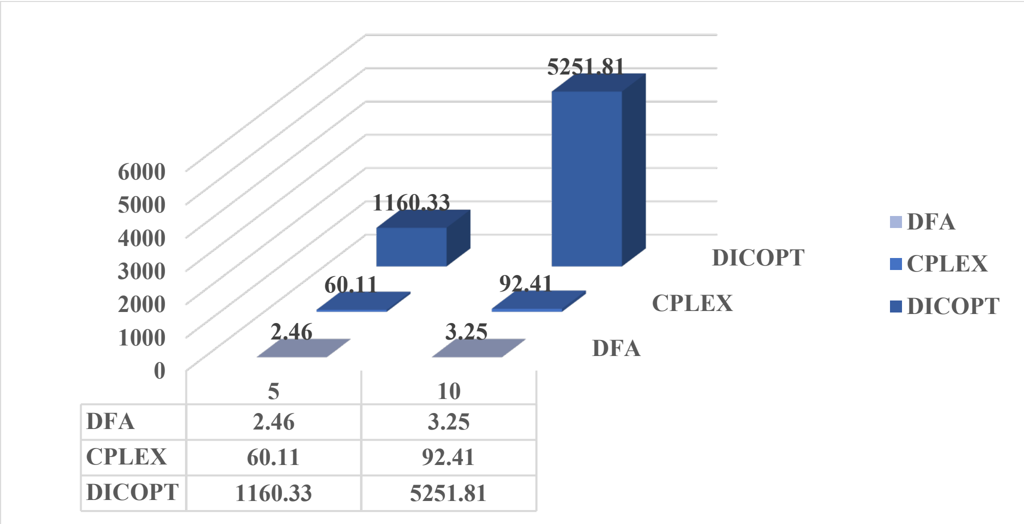

Fig 9. Percentage changes in CPU time for using different approaches

Table 10 and Fig. 8 and Fig. 9 illustrate the significant increase in solution time associated with increasing product numbers, especially in the MINLP solver. As well as being very fast, DFA produces very high-quality solutions. While the Mixed-Integer Programming (MIP) method requires more CPU time compared to the DFA approach, it guarantees the optimal solution, with the CPU time remaining reasonably short. A comprehensive analysis of this matter is presented in Table 11, where the number of product types has been growing.

Table 11
The average and standard deviation values for the differences in objectives and CPU time for cases involving 10 stores, 10 products, 2 brands, and 15, 30, and 70 product types

Product Types	Changes in		CPU seconds		
	Objective (%)	Maximum (%)	DICOPT	CPLEX	DFA
15	0.97 (0.8)	2.1	5251.8 (516.0)	92.41 (29.1)	3.25 (1.3)
30	1.25 (0.8)	7.8	22412.1 (5214.7)	347.2 (79.5)	9.26 (3.2)
60	2.32 (1.3)	12.5	124520.1 (20148.2)	859.4 (101.2)	30.24 (8.2)
All	**1.51 (1.0)**	**7.5**	**50728.0 (8626.3)**	**433.0 (69.9)**	**14.3 (4.2)**

Table 11 and Fig. 10 and Fig. 11 show that MINLP solver CPU time is drastically increased when the number of product types increases, but MIP solver and heuristic methods are less impacted. This scale makes the MINLP solver impractical and the MIP solver and heuristic methods must be used instead. Due to less CPU time, the heuristic method is more effective for large problems. Nonetheless, the MIP solver is still effective for medium-sized problems.

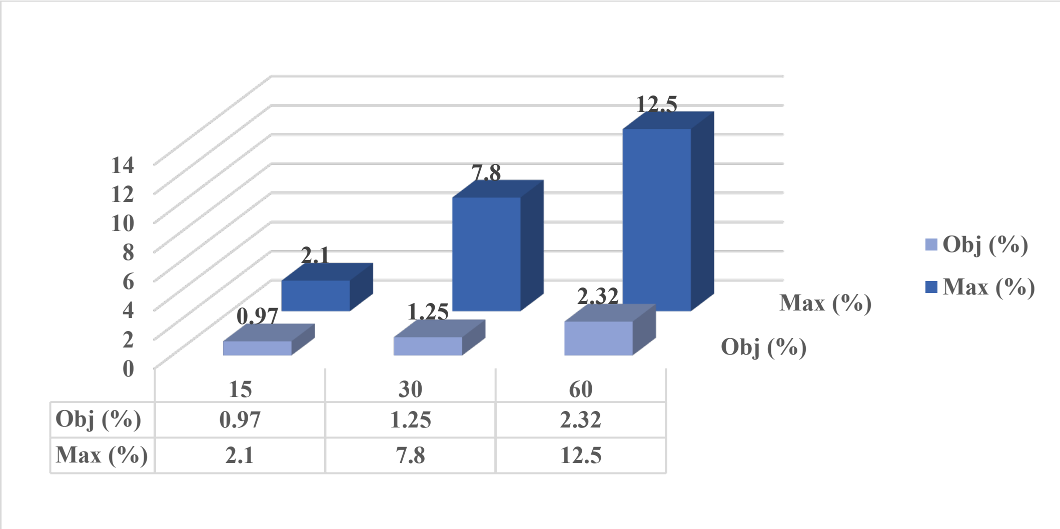

Fig 10. Percentage of Obj and Max changes per product group changes

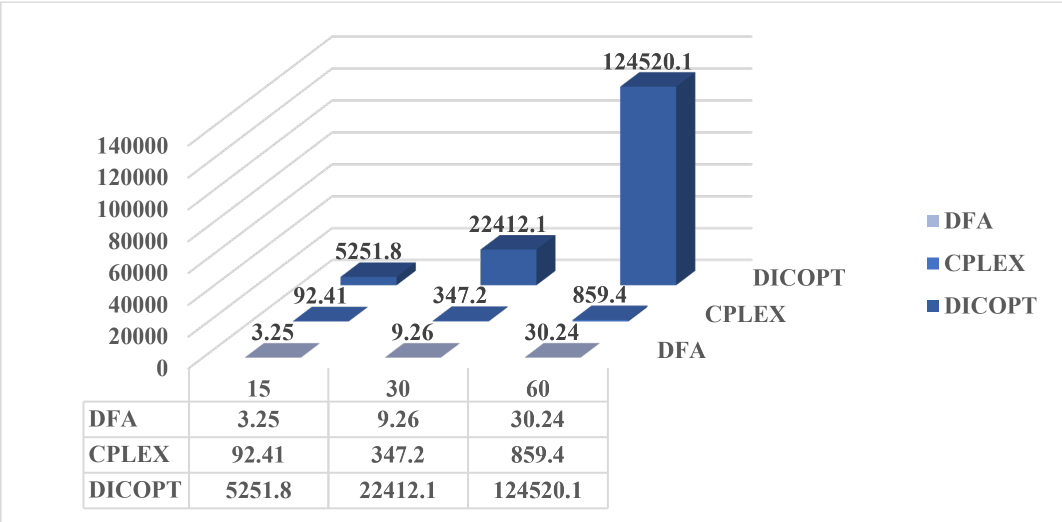

Fig 11. Percentage of CPU changes per product group changes per different approaches

A comparison of MIP solvers and heuristic methods can be seen in Table 12. Based on the results presented in this table, time of CPU for MIP solver significantly grows with the problem size. DFA exhibits lower sensitivity to these differences.

Table 12
Outcomes for the cases involving 70 products types and 10 products

stores	Brands	Changes in		Time of CPU (per second)	
		Objective (%)	Maximum (%)	MIP Solver	Heuristic
10	2	2.3 (1.3)	12.5	859 (101)	30 (8)
	4	2.8 (1.4)	12.6	1025 (351)	46 (10)
20	2	2.9 (1.8)	13.5	1289 (541)	61 (12)
	4	3.5 (1.2)	17.2	2014 (875)	136 (21)
100	2	3.1 (1.0)	15.3	5148 (1025)	214 (36)
	4	3.9 (1.1)	20.9	8941 (3012)	348 (59)

Based on Table 12, CPU time for MIP solver significantly increases with increasing problem size. There is less sensitivity to these changes in the DFA.

The final step involves evaluating the quality of the DFA's solution, irrespective of the problem size. In this study, a representative set of test cases with varying characteristics was used to statistically assess the effectiveness of the DFA method. Similar to earlier tables, 10 problems of varied sizes were created, and every problem was executed 100 repetitions using the DFA algorithm. The outcomes are summarized in Table 11.

Table 13
Outcomes for the Randomly Generated Problems

Case	Stores	Product Types	Products	Brands	Difference in obj (%)
1	10	15	5	2	0.07
2	10	15	5	4	0.47
3	10	60	10	2	1.81
4	20	60	10	4	2.61
5	20	15	5	2	0.73
6	20	30	5	4	1.02
7	100	15	5	2	0.41
8	100	15	10	4	0.35
9	100	30	10	2	1.39
10	100	60	10	4	1.93
Average	**49**	**32**	**8**	**3**	**1.08**
SD	**44**	**21**	**3**	**1**	**0.83**

To evaluate the effectiveness of the DFA algorithm, a t-test (hypothesis testing) is performed. In this context, effectiveness refers to the algorithm's ability to consistently find a high-quality solution. As indicated in Table 13, "Difference in object" is abbreviated as "Diff" and represents the measure of how near a result is to the global optimum.

A test of effectiveness aims to determine whether $\begin{cases} H_0: \mu_{Diff} \leq 1\% \\ H_1: \mu_{Diff} > 1\% \end{cases}$. The objective of this test is to evaluate whether the quality of the obtained solutions is greater than 98%. Eq (19). shows how to calculate the *t* value.

$$t = \frac{\overline{Diff} - 1}{S(Diff)/\sqrt{n}} \tag{19}$$

To evaluate whether the data conforms to a normal distribution, the Kolmogorov-Smirnov (K-S) test is conducted. The resulting test statistic (D) provides a P-value of 0.84497, suggesting that there is no significant departure from a normal distribution. Moreover, the t-test creates a P-value of 0.7675, indicating that the DFA algorithm effectively yields high-quality solutions.

5.4 Case Study

This section focuses on implementing an optimal assortment-promotion model for a chain store situated in Tehran, Iran. We have conducted research for one of the Ofagh Koorosh" chain stores located in the Farmanieh neighborhood. We first divide this neighborhood into different regions and consider all residents in these regions as customers, and the total demand of residents in each region is considered as the demand of that point. For the Farmanieh neighborhood, 18 different regions have been considered, so the number of our customers in this study is 18, and based on the sales history of the past 3 years, we can estimate the demand of each point. Furthermore, considering 154 product groups and 1224 different brands, we will answer the following questions using the model:

- Among the 154 product groups, which group of products has been selected for the store, and what is the optimal set of products among the 1224 different brands in these product groups?
- Considering the existing situations and constraints, what percentage of discount should be considered for each product and brand at present time?

5.4.1 The model Results

Due to the high volume of data in the case study under investigation, we only rely on the final results.

The optimal product assortment includes 151 product groups and 802 brands out of 1224 possible brands in competitor stores (different product groups also have 2 to 7 different brands on the shelves). The optimal discount rates have been calculated for all products and range from 0% to 20% for different products.

5.4.2 Comparing Actual Performance with Model Results

In the two consecutive weeks, a pilot program was tested in three products. The plan involves selecting the store under study as a pilot and solving the proposed model for the three products of tea bags, shampoo, and tomato paste, in order to find the optimal assortment and promotions. For comparing results, we recorded the sales volume and profits for the store based on the previous status between October 24 and October 30 (the first week) as well as the proposed model between October 31 and November 6 (the second week). Due to the possibility that demand may differ between these two consecutive weeks for other reasons, for the pilot store, a similar store (that behaves similarly in terms of sales) has been selected and previous status has been applied to the mentioned products in this store for two weeks. For the pilot and similar stores, Table 14 shows the results.

Table 14
Sales and profit of store and similar store for different brands of tea bags in the first and the second week

Brands	Date	Sales Volume		Profit Value	
		Pilot	Similar	Pilot	Similar
Twinings	Week 1	18	17	144	136
Golestan		12	15	156	195
Shahrzad		14	15	154	165
Ahmad		15	14	165	154
Two gazelles		10	8	100	80
Famila		9	9	90	90
Total		78	78	809	820
Twinings	Week 2	22	19	154	152
Golestan		11	10	132	130
Shahrzad		15	12	150	132
Ahmad		18	15	180	165
Two gazelles		9	9	81	90
Famila		12	14	132	140
Total		87	79	829	809
% Change		12%	1%	2%	-1%

The pilot store has seen an increase in sales and profit in the second week, as shown by Table 14. In similar stores, we have seen an insignificant increase in sales and a decrease in profit. According to the results, the proposed model increased sales and profits in the tea bag product. Table 15 summarizes the results for three products.

Table 15
Total Sales and profit of pilot store and similar store for different products in the first and second week

Products	Date	Sales Volume		Profit Value	
		Pilot	Similar	Pilot	Similar
Black Tea Bag	Week 1	78	78	809	820
Tomato Paste		134	141	1359	1397
Shampoo		99	96	981	962
Total		311	315	3149	3179
Black Tea Bag	Week 2	87	79	829	809
Tomato Paste		149	139	1462	1401
Shampoo		109	96	1102	957
Total		345	314	3393	3167
% Change		10.9%	-0.3%	7.7%	-0.4%

According to Table 15, the similar store (that have not changed their promotion and assortment policy for two consecutive weeks) did not experience an increase in sales or profit, while in the pilot store, the change in promotion discounts and assortment as a result of the use of the proposed model resulted in sales and profit increase which shows the effectiveness of the proposed model in practice.

6 Managerial Insights

- Based on the conducted investigations and the comparison of two different scenarios for determining Promotion and Assortment, it became evident that managers should consider Promotion values across different periods when determining the optimal Assortment. The lack of simultaneous consideration of these two factors will lead to a significant loss of profit.
- Considering the impact of customer inventory holding on product sales in subsequent periods, determining the optimal promotion for a store should be examined with a multi-period horizon. This is because a short-term increase in sales may lead to reduced sales in subsequent periods and an overall reduction in profitability.
- Accurate demand forecasting is essential for determining product assortment, and discounts. Incorrect forecasting leads to nonoptimal solutions. By employing methods for predicting demand, managers aim to accurately forecast future demand with minimal error.
- Success of product promotion relies heavily on understanding the factors that are significant to end-users and the relative importance they assign to each factor. Hence, managers should maintain ongoing communication with end-users to gather their opinions regarding store selection and product offerings. The decision-making process is influenced by the specific product and the economic circumstances of the consumer.
- Enhanced demand for promoted products is a vital aspect of sales promotion. Retailers should communicate with their suppliers to ensure they are aware of the advertised products. It is crucial to consider the possibility that the

increased demand may exceed the supplier's production capacity, leading to inventory shortages for the advertised product. Such shortages can have adverse effects on the credibility of both the supplier and the retailer. Therefore, careful coordination and planning between retailers and suppliers are necessary to avoid such inventory challenges and maintain their credibility in the market.

7 Conclusion

We introduce a new concept in assortment literature in this paper. Choosing the right product portfolio for the stores is also important when deciding how to promote it. Whether a customer chooses products or brands is based on an assortment-oriented approach, meaning they can substitute when their favorite product is unavailable. A model has been devised to optimize promotion and assortment strategies.

In this paper, a nonlinear integer model is proposed. The model is reformulated using mixed-integer linear equations, allowing the use of a standard optimization solver to find the optimal solution for small- and medium-sized problems. Additionally, a heuristic algorithm is developed for solving large-scale problems. The effectiveness of the proposed methods is validated through the solution of several examples and detailed analysis of the results. The findings highlight the importance of simultaneously considering promotion and assortment variables.

There are several possible extensions for this research. One such extension is to explore the location of new facilities. Additionally, various choice models like MNL, Nested, and Mixed MNL can be incorporated into the model. Another potential direction is to investigate the extent of supplier participation in discount rates and analyze different scenarios for their involvement.

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